The Influence of Boundary and Edge-Plasma Modeling in Computations of Axisymmetric Vertical Displacement

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The influence of boundary and edge-plasma modeling in computations of axisymmetric vertical displacement

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Abstract

Three sets of axisymmetric extended-MHD computations explore the sensitivity of vertical-displacement simulations to variations in boundary conditions and thermal conductivity modeling. The parameters represent a modest-sized tokamak. The forced-displacement scenario and computational setup are similar to those used in previous 3D computations [Sovinec and Bunkers, PPCF 61, 024003 (2019)]. For a given wall resistivity, results are most sensitive to variations in boundary and thermal-conduction parameters that affect electron thermal transport. The electrical conductivity depends on electron temperature, so the sensitivities result from their influence on the electrical circuit that includes the open-field halo current. Conditions that lead to hotter, broader halo regions slow the evolution. Sensitivity to the boundary condition on plasma flow-velocity exists when electron thermal conduction is restricted and electron energy loss is convective, which is expected for conditions at the entrance of the magnetic presheath.

1. Introduction

Theoretical advances in macroscopic stability and technological advances in control systems have been important for achieving nearly reactor-grade conditions in tokamak plasma confinement systems. Nonetheless, disruptive loss of macroscopic stability continues to pose substantial risk, particularly as new experiments are brought online and achieve unprecedented levels of plasma thermal energy and releasable magnetic energy. There are many underlying causes for disruptive events, but loss of vertical position control is common, even if vertical instability is not the root cause of the disruption. The ensuing plasma-wall contact can lead to costly hardware damage and loss of operating time. These risks have prompted a variety of theoretical efforts to better characterize vertical displacement events (VDEs) and to mitigate their consequences. The present work focuses on the combined influence of two aspects of extended magnetohydrodynamics (MHD) computations of tokamak VDEs: edge-plasma transport modeling and boundary conditions.

Models for vertical stability and vertical displacement can be as basic as current-carrying loops representing the tokamak electrical discharge. Mukhovatov and Shafranov use this approach to derive a linear stability criterion for large aspect-ratio tori. Linear ideal-plasma analysis with more realistic toroidal representations of the plasma region include elliptically shaped cross sections with parabolic and step-function pressure profiles. Fitzpatrick’s work also considers $n = 1$ stability and non-ideal walls. An ideal core-plasma response, i.e. perfectly conducting, can be justified in conditions of “hot” VDEs, where vertical instability precedes the TQ and the entire confinement region remains intact while displacing and distorting. Here, the separation of time-scales between resistive dissipation, plasma motion, and Alfvénic propagation implies that magnetic topology and force-balance are preserved. Fitzpatrick applies this in Ref. 9 to investigate marginally stable vertical displacement, including the open-field halo current that supports the confinement region after contact with a surface. A more general model developed for these conditions, which also accommodates 3D external kink instability, is
the tokamak-MHD (TMHD) system with a non-ideal wall.\textsuperscript{11} TMHD has been used to investigate the wall-touching kink mode (WTKM)\textsuperscript{12} that Zakharov predicted in Ref. 13. The numerical implementation of this model allows slight force imbalance by including a drag term, i.e.\textsuperscript{13}
\[ \nabla p = \mathbf{J} \times \mathbf{B} - \rho \gamma \mathbf{V}, \]

where \( p \) is the plasma pressure, \( \mathbf{J} \) is current density, \( \mathbf{B} \) is magnetic field, \( \rho \) is mass density, \( \mathbf{V} \) is flow velocity, and \( \gamma \) is the friction coefficient.

For conditions where the TQ precedes vertical displacement, non-ideal modeling of the plasma is more appropriate. A relatively uncomplicated axisymmetric analysis without halo currents avoids modeling thermal energy by effectively assuming that radiative loss maintains a low plasma temperature, hence the use of a prescribed uniform plasma electrical resistivity.\textsuperscript{14} Other time-dependent 2D computational models, such as TSC\textsuperscript{15} and DINA,\textsuperscript{16} address particle and thermal energy transport through flux-surface averaged equations over the confinement region. Resistivity models for the surface-averaged plasma conditions govern the evolution of the current distribution in both implementations. The two codes differ in their solution methods for the force-balance equation, but Ref. 17 shows that their predictions for VDEs in ITER are consistent when the computations do not include differing halo-current and external-structure models.

Computational models that are capable of addressing changes in magnetic topology and its effects on energy and momentum transport solve systems of extended-MHD equations with 3D spatial dependence. Particle density, flow velocity, and temperature are governed by evolution equations with closure relations that approximate plasma behavior. The implementations have applicability to a wide range of magnetized plasma dynamics, which includes tokamak disruption. Although they can reproduce wave propagation with sufficiently small time-steps, the algorithms use implicit and/or semi-implicit methods with large time-steps to address applications with large time-scale separations. Evolution from step to step can be quasi-static with forces remaining nearly balanced. Examples that consider vertical displacement include application of the M3D code to predict toroidal current peaking\textsuperscript{18} and wall forces\textsuperscript{19} in ITER, application of the M3D-C1, JOREK, and NIMROD codes to the destabilization of edge modes through wall contact in NSTX,\textsuperscript{20,21} and the evaluation of relaxation effects in forced-VDE computations with NIMROD.\textsuperscript{22}

Eulerian-based extended-MHD computations solve a single system of equations over open and closed magnetic fields lines. This has the advantage that halo currents are induced, naturally, as magnetic flux surfaces are opened through contact with non-ideal surfaces.\textsuperscript{23} However, representing the entire region inside the first wall with the same system of equations necessitates including cold low-density plasma outside the halo. The strong temperature-dependence of plasma electrical conductivity suppresses current outside the halo, where open-field transport maintains low temperature. A published verification of this approach considers peeling-ballooning modes,\textsuperscript{24} and other tests show that there is less sensitivity to the outer mass density when external modes are slowed by diffusion through a resistive-wall. There is physically relevant sensitivity to the halo-current resistivity in VDE computations, as shown by varying the edge-plasma temperature\textsuperscript{25} and by varying an offset in the Spitzer resistivity relation.\textsuperscript{20} Thus, achieving comprehensive extended-MHD simulations of VDEs will require more detailed modeling of edge plasma and effects from its contact with surfaces.

The work reported here investigates the sensitivity to edge-region modeling of VDEs when applying extended-MHD computations. We compare results of simplified axisymmetric
computations with results obtained with more detailed thermal conduction models and boundary conditions. The simplified edge transport models prescribe parallel and perpendicular thermal diffusivities, and the more comprehensive models are the Braginskii and Ji heat-flux closures, which include magnetization effects. Our boundary-condition investigation considers plasma flow and conductive heat-flux conditions, including magnetic presheath conditions that have been adapted from Ref. 28 and have also been incorporated in the JOREK code. We note that boundary conditions on plasma flow in extended-MHD VDE computations have received attention in the literature, which prompted a comparison that was performed with M3D. Results presented here show that sensitivity to boundary conditions needs to be viewed in light of the full system of equations, and the greatest sensitivities stem from electron heat transport.

Similar to the study reported in Ref. 22, we solve a forced-VDE scenario, which results from the decay of resistive-wall eddy current after an external divertor coil is effectively turned off. We use parameters that represent a modest-sized tokamak with minor radius \( a = 0.25 \text{ m} \), major radius \( R_0 = 0.41 \text{ m} \), magnetic induction \( B_0 = 0.5 \text{ T} \) at \( R = R_0 \), and deuterium plasma with central electron particle density \( n_0 = 10^{19} \text{ m}^{-3} \) and central temperature \( T_0 = 260 \text{ eV} \). Although we are not modeling it per se, such conditions are similar to those expected for the upgraded Pegasus Toroidal Experiment. In our computations with the Braginskii and Ji heat-flux closures, we include an extra factor of 6 so that edge-plasma conditions of \( T = 1 \text{ eV} \) and \( n = 10^{18} \text{ m}^{-3} \) lead to the same perpendicular diffusivity as our computations with prescribed diffusivities. Thermal transport couples to magnetic evolution through the plasma resistivity, which we model with the Spitzer dependence of \( \eta = \eta_0 \left(\frac{T_0}{T}\right)^{3/2} \).

We also consider the influence of boundary conditions on particle density, plasma flow, and electron and ion temperature. The most comprehensive set applied so far models conditions at the magnetic presheath entrance (MPE). It includes the Chodura-Bohm (CB) condition on flow along field-lines and \( \mathbf{E} \times \mathbf{B} \) drift perpendicular to the field-lines. This model is adapted from edge turbulence modeling, and we use just the lowest-order terms with respect to perpendicular spatial scales. The analysis also imposes homogeneous Neumann conditions on electron temperature, i.e. effectively thermally insulating with respect to conduction but allowing electron energy flux from particle loss. Through comparison of computations with different sets of boundary conditions and edge transport models, we find sensitivity where the boundary conditions affect electron temperature profiles, hence magnetic field evolution through the Spitzer resistivity. We also scan wall resistivity and observe results similar to those reported in Refs. 20, 25, and 33.

The rest of the paper is organized as follows. Section 2 describes the equations used to model the VDE and the initial state for the vertical motion. Section 3 shows the results from the most comprehensive base case, and investigates the sensitivities of vertical force, current, and thermal energy to the various boundary conditions, thermal conduction models, and wall resistivity values. Section 4 relates specific observations to general behavior, considers limitations in the modeling, and draws conclusions from this study.

2. Macroscopic Modeling
Fluid-based plasma modeling is applied in this study, similar to the extended-MHD simulations of disruptions that are cited above. The equations are solved with nonlinear initial-value computations, and we use the NIMROD code with the implicit leapfrog time-advance that is described in Ref. 35. Only axisymmetric computations are considered here, but we expect the findings to have implications for 3D simulations. The parameters for the modest-sized tokamak are normalized so that the minor radius and Alfvén time are unity in the numerical computations. The normalized parameters also set magnetic permeability \( \mu_0 \) and effective particle mass \( m \) to unity.

2.1 Fluid-Based Model

The set of equations that is solved for the computational region inside the resistive wall is the compressible visco-resistive MHD model that is extended to include anisotropic thermal and viscous transport:

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{V} + \mathbf{F}_n) = 0 ,
\]

\[
\rho \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi} ,
\]

\[
\frac{n_s}{\gamma - 1} \left( \frac{\partial T_s}{\partial t} + \mathbf{V} \cdot \nabla T_s \right) = -\frac{n_s}{\gamma} \mathbf{V} \cdot \nabla \mathbf{V} - \mathbf{V} \cdot \mathbf{q}_s + Q_s ,
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left( \mathbf{V} \times \mathbf{B} - \eta \mathbf{J} \right) + \kappa_b \nabla \nabla \cdot \mathbf{B} ,
\]

\[
\mu_0 \mathbf{J} = \nabla \times \mathbf{B} ,
\]

where the subscript \( s \) is a species label for ions \( i \) and electrons \( e \). Equation (1) is the continuity equation for particle density \( n = n_e \) with the numerical flux density \( \mathbf{F}_n \) that is discussed in Appendix A. Equations (2) and (3) describe the evolution of plasma center-of-mass flow velocity \( \mathbf{V} \) and species temperature \( T_s \) with \( \rho = mn \) being the mass density for effective particle mass \( m = m_e + m_i / Z \). Here, we take the single-ion-species model to have ion charge state \( Z = 1 \).

Applying quasi-neutrality for low-frequency dynamics, \( n_i = n_e \). The total pressure is then \( p = n \left( T_i + T_e \right) \) with Boltzmann’s constant absorbed in temperature, and \( \gamma = 5/3 \) is the ratio of specific heat capacities. The stress tensor used here has fixed diffusivity coefficients \( \nu \) and \( \nu_b \):

\[
\mathbf{\Pi} = \rho \nu_b \left( \mathbf{b} \cdot \mathbf{W} \right) \hat{b} \left( \mathbf{I} - 3 \mathbf{b} \mathbf{b} \right) - \rho \nu \mathbf{W},
\]

where \( \mathbf{W} = \nabla \mathbf{V} + \nabla \mathbf{V}^T - (2/3) \mathbf{I} \nabla \cdot \mathbf{V} \) is the traceless rate of strain tensor, \( \hat{b} \) is the unit vector along the evolving magnetic field \( \mathbf{B} \), and \( \mathbf{I} \) is the identity tensor. Equation (4) is Faraday’s law with the temperature-dependent electrical resistivity \( \eta \). The last term is used to keep numerical divergence errors small in NIMROD’s spectral-element representation. Its implications for magnetic energy and magnetic helicity are considered for relaxation dynamics in Ref. 36. Equation (5) is Ampère's Law, which relates current density to magnetic field for low-frequency evolution.
We do not apply sources such as loop voltage, a momentum source density, or external heating, so temporal evolution is governed by the model, the system parameters, and the initial state. Also, thermal equilibration and the Ohmic and viscous heating terms represented by \( Q_s \) are not used. However, single-temperature \( \left( T_e = T_i \right) \) computations are also performed to consider the rapid-equilibration limit.

The thermal conduction modeling has either fixed thermal diffusivity coefficients or coefficients from Chapman-Enskog closure. The heat-flux density has the form
\[
\mathbf{q} = -\left( \kappa - \kappa_\perp \right) \hat{\mathbf{b}} \cdot \nabla T - \kappa_\perp \nabla T
\]
with the thermal conductivities and diffusivities related by \( \kappa = \chi n \). We also consider two distinct Chapman-Enskog closure relations, Braginskii’s result and Ji’s \( K = 2 \) approximation.\(^{27}\) Our computations with these models are, respectively, labeled with “B” and “k2”. Both models consider local magnetization measured by \( \chi_s = \Omega_s \tau_s \), where \( \Omega_s = \left| q_s \right| B / m_s \) is the gyrofrequency and \( \tau_s \) is the collisional relaxation time for each species,

\[
\tau_i = \frac{12\pi^{3/2}}{n_i Z^4 e^4} \frac{T_i^{3/2} \epsilon_i^2}{\ln \Lambda_{ii}} \quad \text{and} \quad \tau_e = \frac{12\pi^{3/2}}{2 n_e Z^4 e^4} \frac{T_e^{3/2} \epsilon_e^2}{\ln \Lambda_{ei}}.
\]

The parallel and perpendicular Braginskii thermal diffusivities are

\[
\chi_{s_i} = \frac{T_s \tau_s}{m_s} \frac{\gamma_{0,s}}{\delta_{0,s}} \quad \text{and} \quad \chi_{s_\perp} = \frac{T_s \tau_s}{m_s} \frac{\gamma_{1,s} x_s^2 + \gamma_{0,s}}{x_s^4 + \delta_{1,s} x_s^2 + \delta_{0,s}},
\]

and our computations use the coefficients \( \delta \) and \( \gamma \) defined in Ref. 26 for \( Z = 1 \). The Chapman-Enskog closure by Ji accounts for ion-electron collisions in ion closure relations and corrects known errors\(^{37}\) in Braginskii’s strong-magnetization results for ions. See Ref. 27 for the parallel and perpendicular diffusivities of the \( k2 \) model. An extra factor of 6 is used with both models, so that diffusivities in the nominal edge conditions of \( n = 10^{18} \) m\(^{-3} \) and \( T = 1 \) eV have the same values as our fixed-diffusivity computations. Our two-temperature computations with fixed diffusivities use the normalized values of \( \chi_{s_i} = 5.9 \times 10^{-3} \), \( \chi_{s_\perp} = 1.4 \times 10^{-4} \), \( \chi_{e_i} = 0.20 \), and \( \chi_{e_\perp} = 7.4 \times 10^{-8} \), where the values are normalized by \( a^2 / \tau_A \) with \( \tau_A = R_0 \sqrt{\mu_0 n m_0 / B_0} \) being the Alfvén propagation time. This normalization is approximately \( 3.8 \times 10^5 \) m\(^2\) / s for the parameters of the modest-sized tokamak described in Sect. 1. Single-temperature computations have \( \chi_i = 0.20 \) and \( \chi_\perp = 1.4 \times 10^{-4} \).
The calculations using the Chapman-Enskog closures include limits on the thermal diffusion coefficients. The upper limits on $\chi_\parallel$-values prevent unphysically large parallel diffusion in high-temperature conditions, where plasma is not collisional. The lower limits on $\chi$-values preclude extremely small values to avoid negative values at numerical integration points. For our “$B$” and “$k2$” thermal conduction models, we enforce

$$1.5 \times 10^{-8} \leq \chi_{i,\parallel} \leq 3 \times 10^4, \quad 1.5 \times 10^{-5} \leq \chi_{e,\parallel} \leq 3 \times 10^5,$$

$$\chi_{i,\perp} \geq 1.5 \times 10^{-8}, \quad \chi_{e,\perp} \geq 1.5 \times 10^{-5}.$$

The lower limit on $\chi_{e,\perp}$ prevents these models from reaching the value used in our two-temperature computations with fixed thermal diffusivities. However, a computation labeled “$B_2$” also uses Braginskii thermal conduction but has $\chi_{e,\perp} \geq 1.5 \times 10^{-8}$ to effectively omit this lower limit. The lower limit on $\chi_{i,\perp}$ does not affect the computations.

In all of the computations reported here, the normalized viscous diffusivities are $\nu = 5 \times 10^{-5}$ and $\nu_\parallel = 0.05$. The numerical particle flux density in Eq. (1) is $F_n = -D_n \nabla n + D_h \nabla^2 n$, and our computations have $D_n = 5 \times 10^{-6}$ and $D_h = 10^{-10}$. In addition, lower limits of $0.07 n_0$ and $10^{-4} T_0$ on the particle density and temperature are imposed at the nodes of the numerical expansion to avoid negative values at numerical integration points. The normalized magnetic diffusivity value of $\eta_0/\mu_0 = 10^{-6}$ at the magnetic axis implies magnetic Prandtl and Lundquist numbers of $Pm = \nu_0/\eta_0 = 50$ and $S = \tau_r/\tau_A = 1.3 \times 10^6$, respectively, with the plasma resistive diffusion time $\tau_r = \mu_0 a^2/\eta_0$.

### 2.2. Boundary Conditions

The breakdown of the ion drift approximation at the MPE leads to a set of boundary conditions for fluid moment equations when considering contact with external surfaces. The MPE marks the limit of quasineutrality, and applying boundary conditions there avoids the non-neutral Debye sheath that is adjacent to the wall. The Loizu model uses an ordering for variations in the tangential direction that is perpendicular to $\hat{b}$ and $\hat{n}$, where $\hat{n}$ is normal to the wall. The small parameter of the ordering is $\rho_s/L$, where $\rho_s = c_s/\Omega_i$ is the sound-speed Larmor radius with $c_s = \sqrt{i_i T_i + e_T e_e/m_i}$ and $L$ being the characteristic gradient length along $\hat{b} \times \hat{n}$, which is approximately poloidal in a conventional tokamak. The analysis assumes $T_i \ll T_e$ and uses the isothermal electron condition, $\hat{n} \cdot \nabla T_e = 0$, which is justified by comparison with the electrical potential gradient. At zeroth order in $\rho_s/L$, the model reproduces the CB condition $\hat{b} \cdot \mathbf{V} = \pm c_s$ with the flow pointing outward toward the wall. The
variation in electrical potential along the surface of the wall, which is consistent with tangential-\( \mathbf{E} \) in the resistive wall, leads to perpendicular drifting that can contribute to the outward fluxes of particles and energy. In conditions where electrons are reflected from the wall, the CB condition implies that the electrical current density is limited to the ion saturation current density \( J_{\parallel} = n q_i c_s \), but this effect is not addressed in the present study.

Boundary conditions for the fluid moments are applied at the resistive wall. Section 3 describes the results of a “base” case that sets the boundary flow-velocity to the sum of the CB condition along \( \mathbf{B} \) and the \( \mathbf{E}_{\text{wall}} \times \mathbf{B} \) drift. It also uses an approximation for the isothermal condition \( \mathbf{n} \cdot \nabla T_e \approx 0 \) (see Appendix A), whereas Dirichlet conditions keep ions at the cold \( 10^{-4} T_0 \) limit. For particle density, the base case uses Dirichlet conditions that set \( n = 0.1 n_0 \), which allows particle flux via \( \mathbf{n} \cdot \mathbf{F}_n \neq 0 \). Appendix A shows that it also allows heat flux for both species. We also consider a number of variations, such as applying the advective particle flux boundary condition \( \Gamma_n = \mathbf{n} \cdot n \mathbf{V} \) for the continuity equation, changing the condition on flow velocity, and applying cold Dirichlet conditions on \( T_e \). The boundary condition on magnetic evolution is applied at the outer shell, where \( \mathbf{n} \times \mathbf{E} = \mathbf{0} \). The interface condition for \( \mathbf{B} \) at the resistive wall is discussed in the next subsection.

With the large number of possible combinations, shorthand notation for labeling the computations is helpful. The notation that we use lists symbols for the boundary conditions on \( \mathbf{V}, T_e, T_i, n \), and the thermal-conduction model in that order and separated by slashes. The symbols are defined in Table I. As an example, the shorthand notation for the base case is “VCB+/TeI/TiD/nD/\( \chi B \).”

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Condition Label</th>
<th>Shorthand</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{V} )</td>
<td>Dirichlet</td>
<td>( D )</td>
<td>( \mathbf{V} = \mathbf{0} )</td>
</tr>
<tr>
<td>( \mathbf{V} )</td>
<td>( \mathbf{E} \times \mathbf{B} ) drift</td>
<td>( EB )</td>
<td>( \mathbf{V} = \mathbf{n} \cdot \mathbf{E}_{\text{wall}} \times \mathbf{B} / B^2 )</td>
</tr>
<tr>
<td>( \mathbf{V} )</td>
<td>CB and ( \mathbf{E} \times \mathbf{B} ) drift</td>
<td>( CB+ )</td>
<td>( \mathbf{V} = c_s \mathbf{b} + \mathbf{E}_{\text{wall}} \times \mathbf{B} / B^2 )</td>
</tr>
<tr>
<td>( T_s )</td>
<td>isothermal</td>
<td>( I )</td>
<td>( \mathbf{n} \cdot \nabla T_s \approx 0 )</td>
</tr>
<tr>
<td>( T_s )</td>
<td>Dirichlet</td>
<td>( D )</td>
<td>( T_s = 10^{-4} T_0 )</td>
</tr>
<tr>
<td>( n )</td>
<td>Dirichlet</td>
<td>( D )</td>
<td>( n = 0.1 n_0 )</td>
</tr>
<tr>
<td>( n )</td>
<td>Advective flux</td>
<td>( vf )</td>
<td>( \Gamma_n = \mathbf{n} \cdot n \mathbf{V} )</td>
</tr>
<tr>
<td>( \chi )</td>
<td>Braginskii</td>
<td>( B, B_2 )</td>
<td>See Sect. 2.2</td>
</tr>
<tr>
<td>( \chi )</td>
<td>Ji</td>
<td>( k2 )</td>
<td>See Ref. 27</td>
</tr>
</tbody>
</table>
2.3. Subdomain and Initial Conditions

As illustrated in Figure 1, the computations use separate subdomains for a central plasma region and an outer vacuum region that are interfaced over a thin resistive wall. Each subdomain is meshed for NIMROD’s 2D spectral-element/1D Fourier representation, and just the $n = 0$ Fourier component is used for axisymmetric computations. We solve a magnetic diffusion equation with normalized $\eta/\mu_0 = 100$ to approximate vacuum evolution of $B$ in the outer region while using the same magnetic representation as in the central region. Magnetic evolution in the two subdomains is coupled by the thin resistive-wall relation,

$$\frac{\partial (B \cdot \hat{n})}{\partial t} = -\hat{n} \cdot \nabla \left[ \frac{\eta_w}{\mu_0 \delta_w} \hat{n} \times (B_{vac} - B_{pl}) \right], \quad (10)$$

where $\eta_w$ is the wall resistivity, $\delta_w$ is the thickness of the wall, and the subscripts on $B$ refer to the vacuum and plasma sides of the wall. The factor in brackets on the right side of (10) is the resistive-wall electric field $E_{wall}$ that is noted above for the drift component of the flow boundary condition. In our computations that scan fluid-moment boundary conditions and thermal-conduction modeling, we set $\eta_w/\mu_0 \delta_w = 10^{-3}$. Thus, after normalization resistive diffusion through the wall for tangential scales of order $a$ occurs on the timescale $\tau_w = 10^3$, and $\tau_A << \tau_w << \tau_r$.

---

Figure 1. Schematic of the central plasma and outer vacuum subdomains. The coils are shown with black dots. The red dot indicates the coil that is turned off to induce vertical motion.
The initial state for all of the computations reported here is the up-down symmetric double-null equilibrium shown in Figure 2. It is generated with a free-boundary version of the NIMEQ code\(^3\) on the same 144x144 mesh of bicubic elements that is used for the time-dependent computations. In the closed-flux region, where normalized poloidal flux \( \bar{\psi} \) varies from 0 to 1, the pressure and \( F = R B \phi \) profiles are the quadratic functions \\
\[
\mu_0 P(\bar{\psi}) = P_e + (P_1 - 4P_2\bar{\psi})(1 - \bar{\psi})
\]
and \\
\[
F = F_e + (F_1 - 4F_2\bar{\psi})(1 - \bar{\psi})
\]
with the parameters \\
\[
P_1 = 1 \times 10^{-2}, \quad P_2 = 2 \times 10^{-3}, \quad P_e = 1 \times 10^{-7},
\]
\[
F_1 = 9.25 \times 10^{-2}, \quad F_2 = 1 \times 10^{-2}, \quad F_e = 2.65.
\]
The density profile is set according to \\
\[
n(\bar{\psi}) = \left[ \frac{P(\bar{\psi})}{P(0)} \right]^{1/5}
\]
so that \( n \) initially increases from 0.1 in the open field to 1 at the magnetic axis. As in our 3D study,\(^2\) vertical displacement is forced by effectively removing the upper divertor-coil current for the initial state (Figure 1), which leaves eddy currents in the resistive wall that decay on the \( \tau_w \)-timescale.

Figure 2. Contours of constant plasma pressure (color) and poloidal-flux function (lines) for the initial state of the computations.

3. Results

In presenting our results, we first summarize evolution of the comprehensive \( VCB+T_e/T_i/D/nD/\chi B \) case to show typical forced-VDE evolution. We then consider the influence of changing different boundary conditions and sensitivities when simplifying the thermal transport modeling. Finally, we consider the influence of the wall resistivity in our
forced-VDE scenario. To help distinguish the computations, we use Table II to list the shorthand labels for those in Sections 3.1-3.3 and their key result.

Table II. Summary of computations in Sections 3.1-3.3.

<table>
<thead>
<tr>
<th>Label</th>
<th>Finding</th>
</tr>
</thead>
<tbody>
<tr>
<td>VCB+/T1/T1/DkT/χB</td>
<td>Base: medium halo width and decay time</td>
</tr>
<tr>
<td>VCB+/T1/T1/DkT/χB2</td>
<td>Narrow halo and fast decay</td>
</tr>
<tr>
<td>VCB+/T1/T1/DkT/χk2</td>
<td>Matches base behavior</td>
</tr>
<tr>
<td>VEB/T1/T1/DkT/χB</td>
<td>Wide halo and slow decay</td>
</tr>
<tr>
<td>VEB/T1/T1/DkT/χB2</td>
<td>Medium-narrow halo and moderately fast</td>
</tr>
<tr>
<td>VEB/T1/T1/DkT/χT</td>
<td>Medium-narrow halo and moderately fast</td>
</tr>
<tr>
<td>VEB/T1/T1/DkT/χT2</td>
<td>Matches base behavior</td>
</tr>
<tr>
<td>VCB+/T1/T1/DkT/χF</td>
<td>Medium-wide halo and medium decay</td>
</tr>
<tr>
<td>VEB/T1/T1/DkT/χF</td>
<td>Diffuse and very slow</td>
</tr>
<tr>
<td>VCB+/T1/T1/DkT/χF</td>
<td>Medium-narrow halo and fast</td>
</tr>
<tr>
<td>VEB/T1/T1/DkT/χF</td>
<td>Matches VCB+/T1/T1/DkT/χF</td>
</tr>
</tbody>
</table>

3.1 Base Case

The current-carrying region of plasma moves downward in our computations as the resistive-wall eddy current near the upper divertor coil decays, and the plasma is attracted to the current in the lower divertor coil. The sequence of pressure and poloidal flux shown in Figure 3 helps visualize the global evolution for the base case. Formation of open-field halo current and broadening of the electron-temperature profile, shown in Figures 4a-b, occur during the early phase of the downward motion. The electron species loses heat from the closed-flux region via perpendicular thermal conduction. Where the open field intersects the wall, the isothermal boundary condition \( \hat{n} \cdot \nabla T_e = 0 \) slows parallel electron heat flux density to the rate allowed by convection at the surface. Thus, perpendicular conduction broadens the \( T_e \) profile into the open-field region. It also broadens the electrical conductivity profile, which affects the width of the halo current. In contrast, parallel thermal conduction, together with the cold Dirichlet boundary condition on \( T_i \), keeps \( T_i < T_e \) in the open-field region (Figure 4b-c).

The toroidal current inside the resistive wall and species thermal energy, \( \int \left( \frac{p_s}{\gamma - 1} \right) dVol \), are plotted over the lifetime of the base case VDE in Figure 5. Initially, the current rises slightly and then decays away as the plasma scrapes against the wall. The thermal energy begins a slow decay, and eventually, only cold plasma is left, which allows the final rapid decay of current. Despite the isothermal boundary condition on \( T_e \) that precludes conduction, the convective surface flux (see Appendix A) with the broadened edge region for this species leads to faster loss
of electron thermal energy than ion thermal energy. The increase of current over the first $500r_A$ is from the central plasma region heating electrons in the edge region, which broadens the halo. Considering that the hot core of the plasma tends to conserve poloidal flux, the voltage from $I dL/dt$ with inductance $L$ decreasing from the broadening edge is offset by $L dI/dt$, leading to a positive current bump.

Figure 3. Evolution of the base case VDE, as shown by contours of constant plasma pressure (color) and poloidal-flux function (lines) at times a) 400, b) 1620, c) 3150, and d) 5830. The color scale for pressure is the same as in Figure 2.
Figure 4. Color contours of constant a) $|\mathbf{J}|/B$, b) $T_e$, and c) $T_i$, together with contours of constant poloidal-flux function (lines) at $t = 1620$ of the base case. The blue-red color scale for $|\mathbf{J}|/B$ is saturated and centered at 0 to show the halo current outside the separatrix. The scales for $T_e$ and $T_i$ are the same.

Figure 5. Evolution of a) toroidal plasma current $I$ and b) species thermal energy for the base case.

The total vertical force on the wall is calculated from the Maxwell stress tensor on the resistive wall,

$$F_z = \frac{1}{\mu_0} \oint \hat{n} \cdot \left( \mathbf{B} B - \frac{B^2}{2} \mathbf{I} \right) \cdot \hat{Z},$$

(11)

where the surface integral is over the outer surface of the wall. This relation uses the fact that plasma inertia is small so that the plasma and resistive wall remain in force balance on the VDE timescale. That our computations satisfy this balance has been confirmed, previously. When applied to the base case, we find the total force shown as a function of time in Figure 5a. The
peak of the force occurs just before the final decay, when the plasma current is concentrated near the wall. The surface force density as a function of the poloidal angle \( \theta \) is calculated from

\[
G_z(\theta) = \frac{1}{\mu_0} \left[ \mathbf{n} \cdot \mathbf{B} \mathbf{B} - \mathbf{n} \cdot \frac{B^2}{2} \right] \hat{Z}
\]

with \([ \cdot ]\) indicating the jump across the wall (outside minus inside). The surface force densities for \( t = 0 \) and \( t = 8809 \) are shown in Figure 6a, plotted with respect to the poloidal angle \( \theta \) shown in Figure 6b. The initial concentration results from the imposed transient that replaces the upper divertor coil current with resistive-wall eddy current. Late in time, the force density is localized to the bottom of the vessel, where the plasma touches the wall.

![Figure 6. Plots of a) vertical force density from Eq. (12) and b) geometric definition of the poloidal angle \( \theta = \text{arctan}(Z/R - 1.575) \). The force density is computed for the initial state and during the final decay of the base case.](image)

**3.2. Variations with Boundary Modeling**

We have repeated the axisymmetric forced VDE computation while varying the boundary conditions on temperature, number density, and flow-velocity to gauge their influence. Figures 7 and 8 compare the resulting current and thermal energy evolution with the base case. These figures show that while changing the temperature or velocity boundary conditions with Braginskii thermal conduction can strongly alter global evolution, altering the particle-density boundary condition has little effect. The \( T_e D \) computation with its cold Dirichlet conditions on \( T_e \) loses electron thermal energy faster than the base case, where the loss is through convection. The \( T_e D \) condition keeps the open-field region cold (Figure 9b), which restricts the width of the halo current (Figure 9a). Here, the loss of outer flux surfaces is sufficiently fast that the region inside the last closed flux surface (LCFS) has relatively slow resistive diffusion. This leads to
the layer of large current density inside the LCFS that is analyzed in Ref. Eliminating the CB boundary condition on parallel flow while using the isothermal $\hat{n} \cdot \nabla T_e = 0$ condition (the \textit{VEB} case in Figures 7 and 8) has the opposite effect. Here, the loss of electron thermal energy is slow. The plasma resistive time remains larger, and the resulting electrical conductivity profile is broadened, so the attraction between plasma current and the current in the lower divertor coil is weakened.

![Figure 7](image-url)

Figure 7. Total current in the plasma is shown for various boundary conditions versus the base case in red. Only the changed boundary conditions are shown in the legend, and all cases use Braginskii thermal conduction.
Figure 8. Species thermal energy for electron (dashed) and ions (solid) is shown for various boundary conditions versus the base case in red.

Figure 9. Color contours of constant a) $J_{\|}/B$, b) $T_e$, and c) $T_i$, together with contours of constant poloidal-flux function (lines) at $t = 1120$ of the computation with Dirichlet conditions on $T_e$. The blue-red color scale for $J_{\|}/B$ is saturated at twice the level used in Figure 4a.

The evolution of plasma current and thermal energies displayed in Figures 7 and 8 show that changing the boundary condition on particle density has little effect on the results. Figure 10 compares the density distributions in the vicinity of surface contact, early in the base and $nvf$ computations. The realistic advective boundary condition ($nvf$) allows a smooth outflow. In contrast, the Dirichlet condition with surface-$n$ fixed at 0.1 leads to a thin boundary layer with the small values of $D_n$ and $D_h$ used in our computations. The CB condition on flow-velocity is
applied in both computations, and flows govern the overall particle transport, regardless of the boundary condition on \( n \).

Figure 10. Contours of constant particle density (color) and poloidal-flux function (lines) at \( t = 400 \) from a) the base computation with Dirichlet boundary condition on particle flux (\( nD \)) and b) the computation with the advective (\( mvf \)) boundary condition.

3.3. Variations in Transport Modeling

Our results also show sensitivity with respect to thermal conduction modeling. When perpendicular thermal conduction for the electrons is reduced, relative to the base case, the plasma terminates more quickly. As shown in Figures 11 and 12, this occurs with both the two-temperature, fixed conductivity computation (\( \chi T \)) and the \( \chi B_2 \) Braginskii model with its reduced lower limit on \( \chi_{e,\perp} \). Relative to the base \( \chi B \) model, the effect on the open-field region is similar to imposing the Dirichlet condition on \( T_e \). Although less extreme than the \( T_e D \) case of Section 3.2, the \( \chi B_2 \) model with the isothermal \( \dot{\mathbf{n}} \cdot \nabla T_e = 0 \) boundary condition does not heat as large an edge region as the base case, and a broad halo is not able to form. The agreement between the \( \chi B \) Braginskii model and Ji’s thermal conduction model shown in Figures 11 and 12 is consistent with the fact that the computed conditions do not reach \( T_i >> T_e \), where the two models differ. The influence of the thermal-conduction modeling and boundary conditions leads to quantitative differences in the net vertical forcing. Results from the two Braginskii computations with different lower limits on \( \chi_{e,\perp} \) are compared in Figure 13 as an example.
Figure 11. Total current in the plasma is shown for the different thermal conduction models. The result for the base case is again shown in red.

Figure 12. Total thermal energy for electron (dashed) and ions (solid) is shown for the different thermal conduction models.
Figure 13. Total vertical force on the resistive wall throughout representative computations. The legend indicates what differs from the base case.

Having compared the fixed- and temperature-dependent thermal conduction models, it is worth testing boundary conditions with fixed conductivities. Section 3.1 shows that results can be sensitive to boundary conditions on temperature and flow-velocity, so we vary these conditions with the simplest fixed-conductivity single-temperature model ($\chi_F$). The comparison of plasma current and ion thermal energy (half of the total) in Figures 14 and 15 shows that there is sensitivity to the flow boundary condition when applying the isothermal temperature condition, $\hat{n} \cdot \nabla T = 0$. This behavior is similar to the results of the two-temperature Braginskii model, discussed in Section 3.1. However, the results are not sensitive to the flow boundary condition when applying low-temperature Dirichlet conditions. In fact, results from another single-temperature computation with $\mathbf{V}$ set to zero along the walls (not shown) also match the $TD$ results in Figures 14 and 15.
Figure 14. Plasma current evolution for four single-temperature computations with fixed thermal conductivities ($\chi_F$) with flow and temperature boundary conditions varied. The two $TD$ curves lie on top of each other.

Figure 15. Ion thermal energy evolution for four single-temperature computations with fixed thermal conductivities ($\chi_F$) with flow and temperature boundary conditions varied.
3.4. Sensitivity to Wall Resistivity

Although it has been examined in previous macroscopic computations,\textsuperscript{25,42} we also consider the sensitivity to the wall resistivity. Figures 16 and 17 show the evolution of toroidal current, vertical force, and thermal energies for different values of the resistive-wall parameter $v_{\text{res},w} = \eta_w/\mu_0\delta_w$, which has units of speed. The computations are the same as the $nvf$ case of Section 3.2, otherwise. As expected, the more resistive the wall, the faster the plasma termination, because the stabilizing eddy currents in the wall decay more quickly. For vertically unstable conditions, the VDE growth-rate saturates at a no-wall rate if diffusion through the resistive wall is faster than the plasma response.\textsuperscript{20,25} While vertical motion in the computations presented here results from external forcing and not from a linear instability, decay with the medium $v_{\text{res},w}$-value is closer to that with the largest value than it is to that with the smallest.

There is a clear trend of the vertical force during the final decay decreasing in amplitude with decreasing $v_{\text{res},w}$. The three computations have the same thermal conduction modeling and boundary conditions, so their halo thicknesses are approximately the same. However, decreasing $v_{\text{res},w}$ reduces the rate at which outer flux surfaces are lost. The resulting longer decay leads to a larger, lower-temperature plasma distribution at a given level of current, hence smaller eddy current in the resistive wall and less stress over its outer surface.

Figure 16. Evolution of plasma current and net vertical force on the resistive wall with the resistive-wall parameter varied over a factor of 100.
Figure 17. Evolution of thermal energy for electron (dashed) and ion (solid) species with the resistive-wall parameter varied over a factor of 100.

4. Discussion and Conclusions

Our computations show that the vertical displacement time depends on how wide and hot the halo region is, even in a forced scenario. Although our computations do not scan enough of the parameter space to show scalings, this finding is generally consistent with scalings on vertical drift rate from TSC computations\textsuperscript{43} and on VDE growth rate from analytics.\textsuperscript{9} In extended-MHD computations, symmetric halo current forms without special prescriptions, and it maintains force-balance as the simulated discharge sheds toroidal flux through contact with the resistive wall. For the hot-VDE conditions considered here, the core plasma retains its large electrical conductivity until late in time when the remaining core is so small that its rate of energy transport approaches the rate of displacement. Prior to that point, the distribution of core flux does not change appreciably, and the decay of current in the electrical circuit that includes the halo plasma, sheath effects, and the section of the surface that completes the circuit\textsuperscript{44} regulates the displacement timescale. Simulation parameters that allow a relatively broad, hot halo, such as our base computation, lead to a longer decay time than parameters with either smaller perpendicular electron thermal conductivity or with cold Dirichlet boundary conditions on $T_e$. Parameters that do not affect the halo, such as the boundary condition on $V$ in the single-temperature computations with cold Dirichlet conditions on $T$, do not affect the evolution.

The parameters of the open-field circuit result from a nontrivial combination of effects. The width of the halo current depends on the open-field electrical resistivity, which depends on electron thermal energy transport and surface conditions. The study presented here investigates these sensitivities, but it does not include all related effects that are expected to be important in experiments. Our Chapman-Enskog thermal conductivity computations include effects of turbulent transport only through a prescribed lower limit on diffusivity. Also, the effects of impurity transport, radiation, Ohmic heating, neutral-particle generation and transport, ion
saturation current physics, and secondary electrons are missing. We speculate that the different computations presented here may represent different limits of behavior when at least some of these effects are considered. For example, the isothermal boundary condition on $T_e$ is part of the MPE modeling, but impurity radiation may reduce open-field $T_e$-values to where the results with cold Dirichlet conditions are more representative. Conversely, the ion saturation current limit can lead to smaller current densities at the boundary, which may broaden the halo to maintain the same total current for balancing forces. We note that extension of the CQ due to the formation of a runaway electron beam is not represented in our present computations. The implementation of a reduced model, such as that implemented in other extended-MHD codes, is an area of active development in NIMROD.

Returning to the issue of boundary conditions in extended-MHD computations of VDEs, we emphasize that the vertical displacement timescale and net forcing can be sensitive to the boundary condition on $\mathbf{v}$, but only to the extent that it affects electron thermal transport. Sensitivity of horizontal forcing in 3D computations was reported in Ref. 30, but the influence of other boundary conditions was not considered in that study. Conversely, effects from varying the ratio of prescribed thermal conductivities were considered in Refs. 25 and 33 without relating them to boundary conditions. Since most previous extended-MHD VDE studies have imposed relatively cold Dirichlet conditions on temperature, it stands to reason that they should not have been very sensitive to boundary conditions on $\mathbf{v}$. The temperature-dependent resistivity with a cold boundary allows magnetic flux to penetrate a resistive wall regardless of the plasma surface flow. An obvious point is that the full system of extended-MHD equations with electrical resistivity, thermal conduction, viscosity, and particle diffusion is of higher order in spatial derivatives than more idealized systems, so more boundary conditions are required. Moreover, extended-MHD modeling admits a greater range of behavior than idealized models, and expectations from idealized systems may not be realized.

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**Data Availability Statement**

The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Appendix A**

The numerical particle-flux density $\mathbf{F}_n$ is used to avoid node-scale noise and overshoot in our extended-MHD computations. We have noted that it also admits particle flux through boundaries when $\mathbf{n} \cdot \mathbf{v} = 0$, which precludes particle flux, otherwise. Moreover, although
diffusive $\mathbf{F}_n$ may represent sub-scale particle-flux density, it does not result from projecting moments directly from kinetic equations. Thus, it is worth considering other effects that it can have in VDE computations and corrections that can be applied.

Keeping $\rho \mathbf{V}$ as the mechanical momentum density in the system and using $\nabla \cdot \mathbf{B} = 0$, the conservative form of Eqs. (1-5) includes Eq. (1) and the following:

$$\frac{\partial}{\partial t} (\rho \mathbf{V}) + \nabla \cdot \left[ \rho \mathbf{V} \mathbf{V} - \frac{1}{\mu_0} \mathbf{B} \mathbf{B} + \mathbf{I} \left( p + \frac{B^2}{2\mu_0} \right) + \mathbf{\Pi} \right] = 0 \quad \text{(A1)}$$

$$\frac{\partial}{\partial t} \left( \frac{\rho \nu^2}{2} + \sum_s \frac{nT_s}{(\gamma - 1)} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left\{ \frac{\rho \nu^2}{2} \mathbf{V} + \sum_s \left[ \frac{\gamma nT_s}{(\gamma - 1)} \mathbf{V} + \mathbf{q}_s \right] + \frac{\mathbf{\Pi} \cdot \mathbf{V} + \eta}{\mu_0} \mathbf{E} \times \mathbf{B} \right\} = 0 \quad \text{(A2)}$$

$$\frac{\partial}{\partial t} \mathbf{B} + \nabla \times \mathbf{E} = 0 \quad \text{(A3)}$$

In order to conserve energy without the $\nabla p_e$-term in the Ohm’s law for resistive-MHD, the convective heat-flux densities for electrons and ions uses the same center-of-mass flow velocity. Because NIMROD advances equations for $n$, $\mathbf{V}$, $T_s$, and $\mathbf{B}$, Eqs. (A1-A2) are not used, directly. Instead, using Eq. (1) with the numerical $\mathbf{F}_n$ together with (A1) implies a term that does not appear in (2):

$$\rho \frac{\partial}{\partial t} \mathbf{V} + \rho \mathbf{V} \cdot \nabla \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \mathbf{\Pi} + m \mathbf{V} \cdot \mathbf{F}_n \mathbf{V} \quad \text{(A4)}$$

The last term on the right side of (A4) is a numerical force-density correction that is needed to conserve momentum. Similarly, using Eq. (1) to obtain species temperature equations requires numerical heat densities in order to conserve energy. Removing magnetic and kinetic energy terms from (A2) and separating the electron and ion species produces

$$\frac{n}{\gamma - 1} \left( \frac{\partial}{\partial t} T_e + \mathbf{V} \cdot \nabla T_e \right) = -nT_e \mathbf{V} \cdot \nabla \mathbf{V} - \nabla \cdot \mathbf{q}_e + \eta J^2 + \frac{T_e}{\gamma - 1} \nabla \cdot \mathbf{F}_n \quad \text{and} \quad \text{(A5)}$$

$$\frac{n}{\gamma - 1} \left( \frac{\partial}{\partial t} T_i + \mathbf{V} \cdot \nabla T_i \right) = -nT_i \mathbf{V} \cdot \nabla \mathbf{V} - \nabla \cdot \mathbf{q}_i - \mathbf{\Pi} \cdot \nabla \mathbf{V} + \left( \frac{T_i}{\gamma - 1} - \frac{mV^2}{2} \right) \nabla \cdot \mathbf{F}_n \quad \text{(A6)}$$

The last term in each of (A5) and (A6) is the respective numerical energy-density correction, and we have assigned the kinetic energy contribution to the ion equation. While the momentum-density correction in (A4) is not significant for the relatively slow evolution of VDEs, the energy-density corrections in (A5-A6) can be.

The computations reported in the main part of this article do not include the $\mathbf{F}_n$-related terms in (A4-A6). Considering the importance of electron heat flux in our VDE results, we reexamine
the effective convection. Starting from (A3-A6) with the correction terms removed and assembling the energy-density relation using the continuity equation with $\mathbf{F}_n$ produces:

$$
\frac{\partial}{\partial t} \left( \rho \frac{v^2}{2} + \sum_{s} \frac{nT_s}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left( \rho \frac{v^2}{2} \mathbf{V} + \sum_{s} \left( \frac{\gamma nT_s}{\gamma - 1} \mathbf{V} + \mathbf{q}_s \right) \right) + \Pi \cdot \mathbf{V} 
+ \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} + \left( \sum_{s} \frac{T_s}{\gamma - 1} \right) \mathbf{F}_n = \mathbf{F}_n \cdot \nabla \left( \sum_{s} \frac{T_s}{\gamma - 1} \right)
$$

(A7)

The $\mathbf{F}_n$-term in the square brackets is a numerical energy-flux density, and the term on the right is an unphysical source density. With the small values of $D_n$ and $D_h$ used in our computations, $\mathbf{F}_n$ is only appreciable adjacent to where the displacing tokamak contacts the surface and only when Dirichlet conditions ($nD$) are used for particle density. Near the contact surface at $t = 400$ in the base computation (Fig. 10a), the particle density is small, so $|n\mathbf{V}|$ is small relative to the computation with the advective ($nvf$) boundary condition (Fig. 10b). However, conditions in the boundary layer in the base $nD$ case have density variations of $\Delta n \equiv 0.03$ over $\Delta z \equiv 0.0015$. Thus, the numerical surface flux-density is, approximately,

$$
|\mathbf{F}_n| = D_n \frac{\Delta n}{\Delta z} + D_h \frac{\Delta n}{\Delta z^3} \approx 1 \times 10^{-3}.
$$

In the same location for the computation with the $nvf$ case, the surface flux-density is, approximately, $|\mathbf{n} \cdot \mathbf{b}_n| \approx 7 \times 10^{-4}$. Thus, when using the Dirichlet conditions to fix a small value of surface-$n$, the numerical energy-flux density for electrons in Eq. (A7) compensates for the reduction in physical convection. Also, the isothermal boundary condition on $T_e$ keeps the unphysical source density on the right side of Eq. (7) small. In computations with cold Dirichlet conditions on $T_e$, such as the $T_e D / T_i D$ computation used in Figs. 7-9, electron parallel thermal conduction dominates all convective losses.

We have repeated the base ($nD$ Dirichlet boundary condition on $n$) and $nvf$ (advective condition on $n$) computations including the momentum and energy correction terms in Eqs. (A4-A6). Results on the evolution of current, vertical force, and thermal energies from these computations are compared with results from the base computation in Figure 18. From initiation through $t = 6000$, the largest discrepancy in electron thermal energy between the two computations with corrections is approximately 2%. After $t = 7000$ the $nD$ computation loses electron and ion thermal energy faster than the $nvf$ computation. This late into the evolution, the magnetic axis has displaced more than 70% of the way to the lower surface, and the minor radius is much smaller than it is in the initial configuration. Nonetheless, the two computations show very similar evolution, including their predictions for the maximum vertical force. Apart from the distribution of particle density near the contact surface, other profile information, such as $J || / B$ and temperatures, are quite similar when comparing states at equal levels of current. In comparison with the base case, the computations with correction terms retain more thermal energy in both species, and the maximum predicted force is approximately 33% larger. Early in the evolution, their profiles appear similar to the base case, but their closed-flux temperatures and peak parallel current densities are larger late in time. Nonetheless, the overall behavior of
the computations with and without these corrections is sufficiently similar that the comparisons made without them are valuable.

![Figure 18](image)

**Figure 18.** Comparisons of a) current and vertical force evolution and b) ion and electron thermal energy between the base computation and two with correction terms for density diffusion. The subscript “c” in the figure legends indicates results obtained with the correction terms.

A final point related to boundary conditions in our VDE computations pertains to the NIMROD implementation. With its spectral-element/Fourier representation, the standard implementation solves the differential equations in weak form using the Galerkin method. All of the computations reported here use either homogeneous or inhomogeneous Dirichlet conditions on \( \mathbf{V} \), and these essential conditions are prescribed for the space of trial functions. The same applies to the continuity equation and the temperature equations when they are solved with Dirichlet conditions. Other boundary conditions are imposed through surface integrals. The entire divergence term in the continuity equation (1) is integrated by parts when projecting onto test functions. When using the advective flux condition for the CB and drift flows, the resulting natural conditions enforce

\[
\mathbf{n} \cdot \mathbf{\Gamma} = \mathbf{n} \cdot \left( \pm c_s \mathbf{b} + \frac{1}{B^2} \mathbf{E}_w \times \mathbf{B} \right)
\]

along the inner surface of the resistive wall, where \( \mathbf{\Gamma} \) is the total particle-flux density. However, it is only the conductive heat-flux density in NIMROD’s temperature equations that is integrated by parts. Where it is used, the isothermal condition is enforced by natural conditions for

\[
- (\kappa_{e,\parallel} - \kappa_{e,\perp}) \mathbf{n} \cdot \mathbf{b} \cdot \nabla T_e - \kappa_{e,\perp} \mathbf{n} \cdot \nabla T_e = 0.
\]

To the extent of local anisotropy, the resulting isotherms follow magnetic field-lines to the resistive-wall surface. Moreover, this condition does not preclude convective energy flux.
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