Sound Waves, Thermal Conduction, and the Continuity Equation

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This note discusses an aspect of the system of equations we are presently solving with NIMROD, i. e. fluid equations without the continuity equation. I have parallel motions in mind, so I'm going to ignore magnetic field completely and consider a 1D neutral fluid problem. The objective is to understand what happens to sound waves when we use thermal conduction in our system of equations without continuity.

The fluid moment equations, starting at the lowest and working up, plus closure information are:

$$\frac{\partial n}{\partial t} + \mathbf{V}\cdot\nabla n = -n\nabla\cdot\mathbf{V} \tag{1}$$

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} = -\frac{1}{\rho}\nabla p \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla p = -\gamma p\nabla\cdot\mathbf{V} - (\gamma-1)\nabla\cdot\mathbf{q} \tag{3}$$

$$\mathbf{q} = -\kappa\nabla T \tag{4}$$

$$p = nT \quad . \tag{5}$$

Linearizing the system, and assuming a homogenous equilibrium without flow and uniform thermal conductivity, the moments become

$$\frac{\partial n}{\partial t} = -n_0\nabla\cdot\mathbf{V}$$

$$\frac{\partial \mathbf{V}}{\partial t} = -\frac{1}{\rho_0}\nabla p$$

$$\frac{\partial p}{\partial t} = -\gamma p_0\nabla\cdot\mathbf{V} + \kappa(\gamma-1)\nabla\cdot\left(\frac{1}{n_0}\nabla p - \frac{p_0}{n_0^2}\nabla n\right) \quad .$$

assuming a 1D (in space) *exp*(*ikx-iω t*) dependence, I arrive at the following dispersion relation:

$$1 - \frac{k^2}{\omega^2}\left(\frac{\gamma p_0}{\rho_0}\right) + i\frac{k^2}{\omega}\frac{\kappa(\gamma-1)}{n_0}\left(1 - \frac{k^2}{\omega^2}\frac{p_0}{\rho_0}\right) = 0 \tag{6}$$

From this equation, we can see that if κ goes to zero, we have adiabatic sound waves, and if density goes to infinity, there is just diffusion. It is handy to make the following definitions

$c^2 \equiv \frac{\gamma p_0}{\rho_0}$, defining c as the adiabatic sound speed,

$\tau \equiv \frac{1}{kc}$, the time for the adiabatic wave to propagate $2\pi/k$, and

$\varpi \equiv \omega\tau$, normalized frequency.

Making these substitutions in (6) gives

$$1 - \frac{1}{\varpi^2} + i\frac{D}{\varpi}\left(1 - \frac{1}{\gamma\varpi^2}\right) = 0 \qquad (7)$$

where D is a relative indicator of heat conduction, something like the inverse of the Peclet number, but defined with the adiabatic sound speed and the wavenumber of interest, instead of flow speed and characteristic length.

$$D \equiv \frac{\kappa(\gamma - 1)}{\tau n_0 c^2}$$

From Eq. (7), we can see that large thermal conduction (large D) also gives a sound wave, but the phase speed is modified to reflect the isothermal γ=1.

After fighting with Mathematica, I was able to plot the dispersion relation in the complex ϖ -plane, as shown in Fig. 1. The equation has three roots. The two oscillatory solutions are the sound waves (with positive and negative real parts). They start at ± 1 and end at $\pm\sqrt{\gamma}$ as D goes from zero to infinity. A purely decaying mode also drops down the negative imaginary axis. What's worth noting is that D of order unity introduces moderate dissipation, but away from unity the damping goes away. Thus, the sound-wave springiness is not lost by thermal conduction.

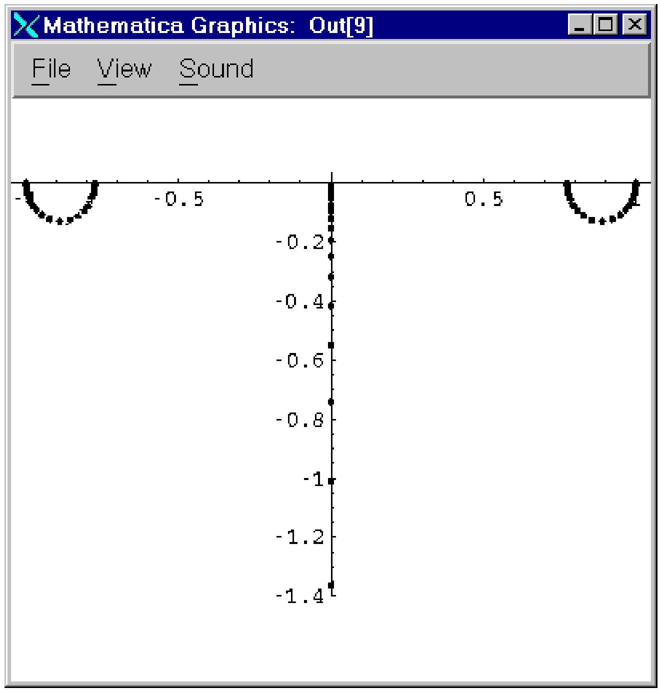

Fig. 1. Roots of Eq. (7) as D is varied logarithmically from 0.01 to 100.

Our modified set of equations without continuity produces a dispersion relation similar to (7), but the continuity term is missing:

$$1-\frac{1}{\varpi^2}+i\frac{D}{\varpi}=0 \tag{8}$$

This alters the roots pretty substantially, unless D is small, as shown in Fig. 2. There are only two roots, and they meet at the negative imaginary axis at D=2, transitioning to purely decaying behavior. Thus, the springiness always present in the complete system is lost unless $D \le 0.2$. For a given set of physical conditions (sound speed and thermal diffusivity), one can use this value of D to determine the scale-length cut-off, below which sound waves don't propagate and the fluid acts infinitely compressible.

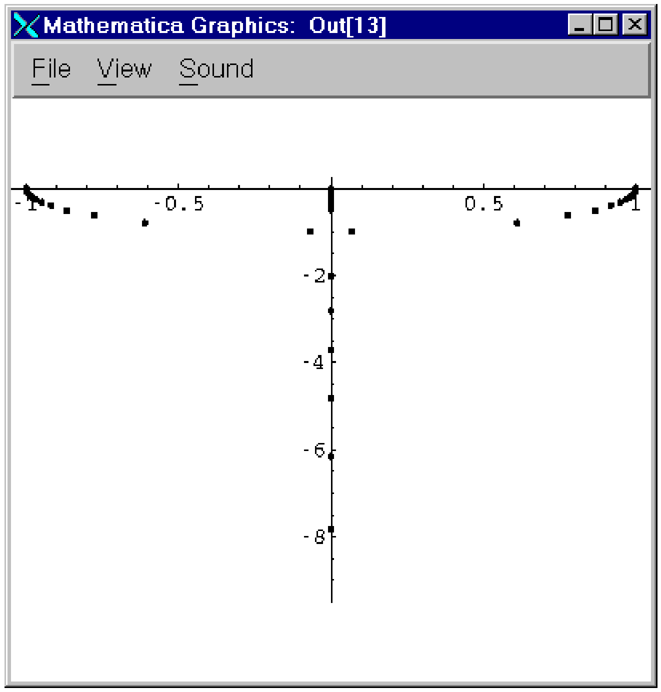

Fig. 2. Roots of Eq. (8) as D is varied from 0.01 to 100.

This issue came up in finite pressure simulations of spheromak sustainment. I wanted to limit compressibility to see what effect that has. However, to avoid having the flow create a pressure hole, I had to increase thermal conduction to the point where all scale lengths in the problem are below the cut-off, anyway. Thus, while the simulation ran, it gave me no information on limiting compressibility.

Having the full set of equations will take care of this "feature," but before that time, we should be aware of the implications. In cases where we expect physical sound wave mixing to flatten pressure anyway, it may not be important.