A Formulation of Fluid Parallel Electron Viscosity for NIMROD

Carl Sovinec, University of Wisconsin-Madison

March 2, 2007, revised April 5, 2007

The parallel part of electron stress is important for macroscopic simulations in at least two or three ways. First, it is the underlying mechanism for neoclassical bootstrap current when balanced with other forces acting on the electrons. Second, when two-fluid reconnection occurs, resistivity is typically too small to account for the rate of change of reconnected magnetic flux. Electron viscosity, possibly from turbulence, is often invoked as the topology-changing effect. Third (and related to the second), a spatial damping on the electron fluid is needed in nonlinear simulations to avoid oscillations at the smallest scale supported by the representation. Resistivity is just a drag on the electron fluid, so it is not very effective at damping electron-fluid oscillations.

Implementing an electron viscosity in NIMROD is not trivial, because it requires us to solve an auxiliary field in the magnetic advance. In terms of the charge-current density (**J**) and the center-of-mass flow velocity (**V**), the electron velocity is

$$\mathbf{V}_e = \mathbf{V} - \left(\frac{1}{1 + Z m_e / m_i} \right) \frac{1}{ne} \mathbf{J} \cong \mathbf{V} - \frac{1}{ne} \mathbf{J} \quad , \tag{1}$$

where n is the electron number density, and Z is the ion charge in units of the elementary charge, e. The fluid form of the electron stress ($\underline{\Pi}$) involves first derivatives of $\mathbf{V}_e$, so it needs second derivatives of magnetic field (**B**), since $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$. In addition, the divergence of the electron stress is proportional to a force that contributes to the electric field,

$$\mathbf{E} = \eta \mathbf{J} - \mathbf{V} \times \mathbf{B} + \frac{1}{ne} \left[\mathbf{J} \times \mathbf{B} - \nabla p_e \right] - \frac{1}{ne} \nabla \cdot \underline{\Pi} \quad . \tag{2}$$

We need the curl of this in the magnetic advance (Faraday's law), so the term has fourth-order derivatives of **B**—explaining why simplified versions are considered to be hyper-resistivity. In the weak form, we can integrate half of the derivatives by parts. Without an auxiliary field, that would still require products of second-order derivatives of **B** and second-order derivatives of the test functions. Our basis functions do not have continuous derivatives (for good reason), so these terms lead to undefined products of delta functions in the volume integrals. Defining an auxiliary field to represent stress or part of the stress leads to products of first derivatives only.

This topic of this note is the choice of the auxiliary field for the electron stress term. If we simply choose the six unique components of $\underline{\Pi}$, we would need to solve a 9-vector system to advance **B**. That would be horribly slow in a computation. The correct choice of a parallel component of $\underline{\Pi}$ will let us solve a 4-vector system, which will not be a significant computational penalty for the additional physical effects. We focus on the collisional (Braginskii) form of the electron stress. However, the expectation is that the same implicit operator used for the collisional stress will be useful as a semi-implicit operator for the integral closures of E. Held.

The collisional form of the parallel electron stress, written in dyad notation (from notes by Jim Callen), is

$$\underline{\Pi} = -\frac{3}{2}\eta_0\left(\hat{\mathbf{b}}\cdot\mathbf{W}\cdot\hat{\mathbf{b}}\right)\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I}\right) \tag{3}$$

where $\hat{\mathbf{b}}$ is the unit direction vector for the magnetic field, and the viscosity coefficient η_0 depends strongly on temperature but only logarithmically on number density. The symmetric rate-of-strain tensor **W** is

$$\mathbf{W} = \nabla\mathbf{V}_e + \nabla\mathbf{V}_e^{\,T} - \frac{2}{3}\left(\nabla\cdot\mathbf{V}_e\right)\mathbf{I} \quad , \tag{4}$$

so Eq. (3) can be written as

$$\underline{\Pi} = -3\eta_0\left(\hat{\mathbf{b}}\cdot\nabla\mathbf{V}_e\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\mathbf{V}_e\right)\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I}\right) \quad . \tag{5}$$

As apparent from Eq. (1), this includes contributions from **V** in addition to those from **J**. In NIMROD's leapfrog-based advance, the contributions from **V** will lead to explicit terms in the magnetic advance, apart from the fact that $\hat{\mathbf{b}}$ also changes in time. Ignoring the changes in $\hat{\mathbf{b}}$ for now, the stress tensor can be separated into relatively compact implicit and explicit contributions, $\Delta\underline{\Pi}+\underline{\Pi}$. The implicit part results from changes in $\mathbf{V}_e$ due to changes in magnetic field,

$$\Delta\underline{\Pi} \equiv \frac{3\eta_0\theta}{e\mu_0}\left[\hat{\mathbf{b}}\cdot\nabla\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right]\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I}\right) \tag{6}$$

and

$$\Delta\Pi_{||} \equiv \frac{3\eta_0\theta}{e\mu_0}\left[\hat{\mathbf{b}}\cdot\nabla\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right] \tag{7}$$

where θ is a numerical centering parameter. The explicit part results from **V**, which is already time-centered from the temporal staggering of **V** and **B** in NIMROD, and from the old current density, $\mathbf{J}^{\mathrm{n}}$:

$$\Pi_{||} \equiv -3\eta_0\left[\hat{\mathbf{b}}\cdot\nabla\left(\mathbf{V} - \frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\mathbf{V} - \frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\right] \tag{8}$$

To see the effect on the operator for the magnetic advance, lump all other contributions to the electric field into $\mathbf{E}_{other}$. The magnetic advance then appears as

$$\Delta\mathbf{B} - \Delta t\nabla\times\left(\frac{1}{ne}\nabla\cdot\Delta\underline{\Pi}\right) = \Delta t\nabla\times\left(\frac{1}{ne}\nabla\cdot\underline{\Pi}\right) - \Delta t\nabla\times\mathbf{E}_{other} \tag{9}$$

This is equivalent to the following system

$$\Delta\mathbf{B} - \nabla\times\left\{\frac{1}{n}\nabla\cdot\left[\nu\, f\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I}\right)\right]\right\} = -\Delta t\nabla\times\mathbf{E}_{other} \tag{10a}$$

$$
\begin{aligned}
&f - \nu\left[\hat{\mathbf{b}}\cdot\nabla\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right] = \\
&\qquad -\sqrt{\frac{3\eta_0\mu_0\Delta t}{\theta}}\left\{\hat{\mathbf{b}}\cdot\nabla\left(\mathbf{V} - \frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\mathbf{V} - \frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\right\}
\end{aligned}
\tag{10b}
$$

with the coefficient

$$
\nu \equiv \frac{\sqrt{3\Delta t\,\eta_0\theta}}{e\sqrt{\mu_0}}
$$

Multiplying the left side of (10a) by a vector test function **A**, the left side is

$$
\begin{aligned}
&\mathbf{A}\cdot\Delta\mathbf{B} + \nabla\cdot\left\{\mathbf{A}\times\left[\frac{1}{n}\nabla\cdot\nu f\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I}\right)\right]\right\} - \nabla\times\mathbf{A}\cdot\left[\frac{1}{n}\nabla\cdot\nu f\left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\mathbf{I}\right)\right] \\
&= \mathbf{A}\cdot\Delta\mathbf{B} + \nabla\cdot\{\ \} - \frac{\nabla\times\mathbf{A}}{n}\cdot\left[\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla(\nu f) - \frac{1}{3}\nabla(\nu f) + \nu f\left(\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}} + \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right)\right]
\end{aligned}
\tag{11}
$$

where the total divergence is not repeated on the second line. Multiplying (10b) by a scalar test function *g*, the left side is

$$
\begin{aligned}
&gf - g\nu\left[\hat{\mathbf{b}}\cdot\nabla\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right] \\
&= gf - g\nu\left\{\hat{\mathbf{b}}\cdot\left[\nabla\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\cdot\hat{\mathbf{b}}\right) - \left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\times\nabla\times\hat{\mathbf{b}} - \left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\nabla\hat{\mathbf{b}}\right] - \frac{1}{3}\nabla\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right\} \\
&= gf - \nabla\cdot\left\{g\nu\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\cdot\hat{\mathbf{b}}\right)\hat{\mathbf{b}} - \frac{g\nu}{3}\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right\} + \left(\frac{\nabla\times\Delta\mathbf{B}}{n}\cdot\hat{\mathbf{b}}\right)\nabla\cdot\left(g\nu\hat{\mathbf{b}}\right) \\
&\qquad + g\nu\hat{\mathbf{b}}\cdot\left[\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\times\nabla\times\hat{\mathbf{b}} + \left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\nabla\hat{\mathbf{b}}\right] - \frac{1}{3}\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\nabla(g\nu) \\
&= gf - \nabla\cdot\{\ \} + \left(\hat{\mathbf{b}}\cdot\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\left[\hat{\mathbf{b}}\cdot\nabla(g\nu) + g\nu\nabla\cdot\hat{\mathbf{b}}\right] + g\nu\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right) - \frac{1}{3}\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\nabla(g\nu)
\end{aligned}
$$

or

$$
\begin{aligned}
&gf - g\nu\left[\hat{\mathbf{b}}\cdot\nabla\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\hat{\mathbf{b}} - \frac{1}{3}\nabla\cdot\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right] \\
&\quad = gf - \nabla\cdot\{\ \} + \frac{\nabla\times\Delta\mathbf{B}}{n}\cdot\left[\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla(g\nu) - \frac{1}{3}\nabla(g\nu) + g\nu\left(\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}} + \hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right)\right]
\end{aligned}
\quad . \tag{12}
$$

The same integration-by-parts steps are needed for the right side of (10b), because $\mathbf{J}^n$ itself is the curl of an expanded field, and in general, it is usually best to make the left and right sides consistent. The weak form of system (10a-b) is then

$$\int_R d\mathbf{x}\left\{\mathbf{A}\cdot\Delta\mathbf{B}-\frac{\nabla\times\mathbf{A}}{n}\cdot\left[\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla(\nu f)+\nu f\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}+\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}}\right)-\frac{1}{3}\nabla(\nu f)\right]\right\}$$

$$+\int_{\partial R} d\mathbf{S}\cdot\left\{\mathbf{A}\times\left[\frac{1}{n}\nabla\cdot\nu f\left(\hat{\mathbf{b}}\hat{\mathbf{b}}-\frac{1}{3}\mathbf{I}\right)\right]\right\}=-\Delta t\int_R d\mathbf{x}\,\mathbf{E}_{other}\cdot\nabla\times\mathbf{A}+\Delta t\int_{\partial R} d\mathbf{S}\cdot\mathbf{A}\times\mathbf{E}_{other} \tag{13a}$$

$$\int_R d\mathbf{x}\left\{gf+\frac{\nabla\times\Delta\mathbf{B}}{n}\cdot\left[\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla(g\nu)+g\nu\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}+\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}}\right)-\frac{1}{3}\nabla(g\nu)\right]\right\}$$

$$-\int_{\partial R} d\mathbf{S}\cdot\left\{g\nu\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}-\frac{g\nu}{3}\left(\frac{\nabla\times\Delta\mathbf{B}}{n}\right)\right\}$$

$$=\int_R d\mathbf{x}\left\{\frac{\mu_0 e}{\theta}\left(\mathbf{V}-\frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\cdot\left[\hat{\mathbf{b}}\hat{\mathbf{b}}\cdot\nabla(g\nu)+g\nu\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}+\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}}\right)-\frac{1}{3}\nabla(g\nu)\right]\right\} \tag{13b}$$

$$-\int_{\partial R} d\mathbf{S}\cdot\left\{\frac{\mu_0 e g\nu}{\theta}\left(\mathbf{V}-\frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\cdot\hat{\mathbf{b}}\hat{\mathbf{b}}-\frac{\mu_0 e g\nu}{3\theta}\left(\mathbf{V}-\frac{1}{ne}\mathbf{J}^{\mathrm{n}}\right)\right\}$$

The symmetry of the volumetric terms is apparent in the system (13a-b) and our implicit operator for electron viscosity can be constructed from this system with f as a scalar auxiliary field. The surface term in (13a) is part of the $\mathbf{E}_{tang}$ surface term, so it should not need further attention. Dropping the surface terms in (13b) may require the viscous coefficient to go to zero (smoothly) approaching the wall.

While all of the integrations-by-parts have been performed in deriving (13a-b), more product rules will be needed in NIMROD. The gradients on νf and νg need to be carried out on each factor separately. Similarly, $\hat{\mathbf{b}}=\mathbf{B}/B$, and only the $\mathbf{B}$-vector itself is an expanded field (not its magnitude), so the best way to evaluate the derivatives of $\hat{\mathbf{b}}$ may be to use

$$\nabla B=\frac{\nabla\left(B^2\right)}{2B}=\frac{1}{B}\left[\mathbf{B}\times(\nabla\times\mathbf{B})+\mathbf{B}\cdot\nabla\mathbf{B}\right]$$

in

$$\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}+\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}}=\frac{\nabla B}{B}-\frac{1}{B^2}\mathbf{B}\times(\nabla\times\mathbf{B})-\frac{2}{B^3}\mathbf{B}\mathbf{B}\cdot\nabla B$$

to arrive at

$$\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}+\hat{\mathbf{b}}\nabla\cdot\hat{\mathbf{b}}=\frac{1}{B^2}\mathbf{B}\cdot\nabla\mathbf{B}-\frac{2}{B^4}\mathbf{B}\left[\mathbf{B}\cdot(\nabla\mathbf{B})\cdot\mathbf{B}\right]\ . \tag{14}$$

Of course, we will also need to deal with the separate steady state and with $\Delta\mathbf{B}$ arising from $\hat{\mathbf{b}}$ in the Newton-like terms, so the full nonlinear implementation will be very complicated.