Effects of a weakly 3-D equilibrium on ideal MHD instabilities

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The effect of a small three-dimensional equilibrium distortion on an otherwise axisymmetric configuration is shown to be destabilizing to ideal MHD modes. The calculations assumes that the 3-D fields are weak and that shielding physics is present so that no islands appear in the resulting equilibrium. An eigenfunction that has coupled harmonics of different toroidal mode number is constructed using a perturbation approach. The theory is applied to the case of tokamak H-modes with shielded resonant magnetic perturbations (RMPs) present indicating RMPs can be destabilizing to intermediate-n peeling-ballooning modes.
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I. Introduction

The application of small three-dimensional magnetic perturbations to an otherwise axisymmetric configuration has a number of implications for confinement physics [1, 2]. In particular, the use of external 3-D resonant magnetic perturbations (RMPs) whose resonant surfaces lie in the edge region of H-mode tokamak plasmas can under certain conditions suppress the appearance of edge localized modes (ELMs) [3–5]. RMPs are thought to affect transport properties in the edge region, thereby altering profiles that are the source for ideal MHD instabilities associated with ELMs. Indeed, much of the work interpreting the role of RMPs on tokamak performance is geared towards providing models for the transport properties. However, 3-D fields can also directly alter ideal MHD stability boundaries and thereby directly modify ELM physics. Quantifying the effect of a weak 3-D equilibrium distortion on the ideal MHD spectrum is the primary motivation for the following calculation.

The original intention for using RMPs as a method for ELM suppression was based on the expectation that RMPs of sufficient magnitude would produce overlapping magnetic islands that would greatly enhance edge plasma transport [6]. Associated with this enhanced transport would be the reduction of edge localized pressure gradient and currents that drive intermediate wavelength peeling-ballooning modes thought to be responsible for ELM onset [7–10]. However, plasma flows in the edge region may be of sufficient magnitude to prevent RMPs from penetrating and producing magnetic islands [11–14]. If this is physics is operable, a different mechanism is required to explain the observed changes in the edge plasma transport [15–17]. A number of theoretical models in this area are under development [18–21].

While magnetic islands may be suppressed, applied non-axisymmetric coils do induce 3-D variations to the plasma shape [22]. In Ref. [19], it is shown that small 3-D distortions of the MHD equilibrium can adversely affect the local mode stability of the tokamak edge. In particular, infinite-n ideal MHD ballooning stability boundaries are strongly modified in the presence of a 3-D field when a rational value of q is approached. The physics of this result is due to changes in the Pfirsch-Schluter current properties associated with the 3-D rippling of the magnetic surfaces [19, 20]. This strongly modifies the local shear in a manner that is destabilizing to ideal ballooning modes. This effect is present even when strong shielding physics prevents the formation of islands. In principle, any localized instability can be analogously affected. Hence, 3-D distortions to the equilibrium can directly alter micro-instabilities and their associated anomalous transport properties.

The intermediate-n stability of ideal MHD modes thought to be responsible for ELM onset requires profile information throughout the pedestal region. For this reason, the local mode analysis of Ref. [19] is not directly applicable. A perturbative approach to calculating macrostability is pursued here whereby 3-D equilibria are constructed assuming small 3-D distortions away from an axisymmetric equilibrium. Moreover, shielding physics is assumed to be sufficiently strong so as to prevent the formation of large magnetic islands that complicate the ideal MHD stability properties. The procedure allows one to construct perturbed eigenfunctions that contain coupled harmonics with different toroidal mode numbers. The theory shows that generally in the limit of small distortions, 3-D fields are destabilizing to the most unstable/least stable ideal MHD perturbation. Therefore, RMPs can be destabilizing to peeling-ballooning modes.

The approach employed here does not require a particular procedure for constructing the 3-D MHD equilibrium. Indeed, how to model the 'correct' plasma response to applied 3-D fields is an important topic for which multiple approaches are being sought [23]. While stability tools capable of utilizing 3-D equilibrium are available in the stellarator community [24, 25], it is not clear that equilibria used in these studies are appropriate for the RMP problem. In particular, shielding physics (which is also thought to be operative in stellarators as well [26, 27]) is not generally accounted for in 3-D MHD equilibrium codes.

In the following section, a perturbation approach is developed to describe how small 3-D equilibrium distortions modify the ideal MHD stability properties. In Section III, the theory is specialized for applications to the effects of RMPs on MHD stability. An example calculation of potential relevance to the RMP problem is presented where the dominant 3-D modification to the equilibrium is due to localized eddy currents present at rational surfaces when plasma shielding physics is effective. A discussion of this work is presented in the final section.

II. MHD eigenvalues

Consider an MHD equilibrium that can be described as the sum of an axisymmetric solution and a weak 3-D correction. Further, assume that this equilibrium is topologically toroidal. We are now interested in the ideal MHD stability properties of this system. The ideal MHD force operator can be separated as follows

$$\rho\frac{\partial^2\vec{\xi}}{\partial t^2} = \mathbf{F}_0(\vec{\xi}) + \delta\mathbf{F}(\vec{\xi}), \tag{1}$$

where $\mathbf{F}_0$ is the force operator for the axiysmmetric equilibrium and $\delta\mathbf{F}$ is associated with the 3-D correction (its precise form is given in Appendix A). A calculation of the ideal MHD spectrum as a perturbation series in the strength of the 3-D distortion is sought.

Each eigenvector for $\mathbf{F}_0$ has a unique toroidal mode number n. For each n, there is a set of ideal MHD eigenvalues and eigenvectors. The most unstable (or least stable) of these values is generally of interest. The eigenvalues and eigenfunctions are denoted $\omega^2_{\mathbf{n}0}$ and $\vec{\xi}_{\mathbf{n}0}$ consistent with

$$\omega^2_{\mathbf{n}0}\rho\vec{\xi}_{\mathbf{n}0} = -\mathbf{F}_0(\vec{\xi}_{\mathbf{n}0}), \tag{2}$$

with $\vec{\xi}_{\mathbf{n}0} \sim e^{-in\zeta}$ for toroidal angle ζ (i. e., the vector $\mathbf{n}$ labels eigenmodes with toroidal mode number n). To construct the eigenfunctions of the full system (including the 3-D distortion), the eigenfunctions of the axisymmetric equilibrium can be used as an orthogonal basis set

$$\vec{\xi}_{\mathbf{n}} = \sum_{\mathbf{k}} a_{\mathbf{nk}}\vec{\xi}_{\mathbf{k}0}, \tag{3}$$

where each $\vec{\xi}_{\mathbf{k}0}$ is normalized by $\int d^3\mathbf{x}\rho|\vec{\xi}_{\mathbf{k}0}|^2 = 1$, ρ is the mass density and the integral is over the entire plasma volume. Inserting this form in Eq. (1) using Eq. (2), we find

$$\rho\omega^2_{\mathbf{n}}\sum_{\mathbf{k}} a_{\mathbf{nk}}\vec{\xi}_{\mathbf{k}0} = \rho\sum_{\mathbf{k}}\omega^2_{\mathbf{k}0}a_{\mathbf{nk}}\vec{\xi}_{\mathbf{k}0} - \sum_{\mathbf{k}} a_{\mathbf{nk}}\delta\mathbf{F}(\vec{\xi}_{\mathbf{k}0}). \tag{4}$$

Projecting this equation along $\vec{\xi}^*_{\mathbf{m}0}$ and integrating over the plasma volume produces the result

$$\omega^2_{\mathbf{n}}a_{\mathbf{nm}} = \omega^2_{\mathbf{m}0}a_{\mathbf{nm}} + \sum_{\mathbf{k}} a_{\mathbf{nk}}V_{\mathbf{mk}}, \tag{5}$$

where the matrix element $V_{\mathbf{mk}}$ is defined by

$$V_{\mathbf{mk}} = -\int d^3\mathbf{x}\vec{\xi}^*_{\mathbf{m}0}\delta\mathbf{F}(\vec{\xi}_{\mathbf{k}0}). \tag{6}$$

Owing to the Hermitian property, $V^*_{\mathbf{km}} = V_{\mathbf{mk}}$. The precise form for the matrix elements are given in Appendix A.

To make further progress, the coupling coefficients are assumed to be weak for $\mathbf{n} \neq \mathbf{m}$ in the series. Moreover, we anticipate the result that $a_{\mathbf{nn}} \approx 1$ in the weak perturbation limit. Hence, an approximate solution to Eq. (5) is

$$a_{\mathbf{nm}} = \frac{V_{\mathbf{mn}}}{\omega^2_{\mathbf{n}} - \omega^2_{\mathbf{m}0}}. \tag{7}$$

In this limit, the modified eigenvector contains a dominant toroidal harmonic and weakly coupled toroidal sideband harmonics due to the 3-D equilibrium distortion.

$$\vec{\xi}_{\mathbf{n}} = \vec{\xi}_{\mathbf{n}0} + \sum_{\mathbf{k}}\frac{V_{\mathbf{kn}}}{\omega^2_{\mathbf{n}} - \omega^2_{\mathbf{k}0}}\vec{\xi}_{\mathbf{k}0}. \tag{8}$$

Using Eq. (7) for the coupling coefficients, we find from Eq. (5) for the special case $\mathbf{n} = \mathbf{m}$

$$\omega^2_{\mathbf{n}} = \omega^2_{\mathbf{n}0} + V_{\mathbf{nn}} + \sum_{\mathbf{k}}\frac{|V_{\mathbf{nk}}|^2}{\omega^2_{\mathbf{n}} - \omega^2_{\mathbf{k}0}}. \tag{9}$$

This is the solution to the eigenvalue equation to second order in the perturbation amplitude. Noting that the small correction to the force operator is intended to be purely three-dimensional, we anticipate the diagonal matrix element to vanish, $V_{\mathbf{nn}} = 0$. In this case, the approximate eigenvalue expression is given by

$$\omega^2_{\mathbf{n}} = \omega^2_{\mathbf{n}0} + \sum_{\mathbf{k}}\frac{|V_{\mathbf{nk}}|^2}{\omega^2_{\mathbf{n}0} - \omega^2_{\mathbf{k}0}}. \tag{10}$$

What is usually of interest is the stability properties of the most unstable (or least stable) MHD mode. For this particular mode, the denominator of the sum term in Eq. (10) is always negative ($\omega^2_{\mathbf{n}0} < \omega^2_{\mathbf{k}0}$). Hence, the correction term in Eq. (10) is manifestly negative and therefore destabilizing for the most unstable mode of the axisymmetric configuration. This behavior is illustrated in Figure 1, where the spectrum of an axisymmetric equilibrium is compared against the MHD stability of the associated equilibrium weakly perturbed by a 3-D distortion (details for how this spectrum is calculated are given in the following section). For the axisymmetric equilibrium, the most unstable mode corresponds to $n = 10$. In the presence of 3-D fields, toroidal mode number is no longer a good quantum number. However, in the weakly 3-D equilibrium, the eigenmode whose mode structure is dominated by $n = 10$ has a larger growth due to the 3-D distortion. For this case, the 3-D distortion has toroidal mode number $N = 3$ which couples the dominant $n = 10$ mode to the $n = 7$ and $n = 13$ harmonics. In addition to the 3-D coupling enhancing the growth rate of the $n = 10$ mode, the $n = 10$ harmonic also provides a stabilizing contribution to the $n = 7$ and $n = 13$ modes. The result is a

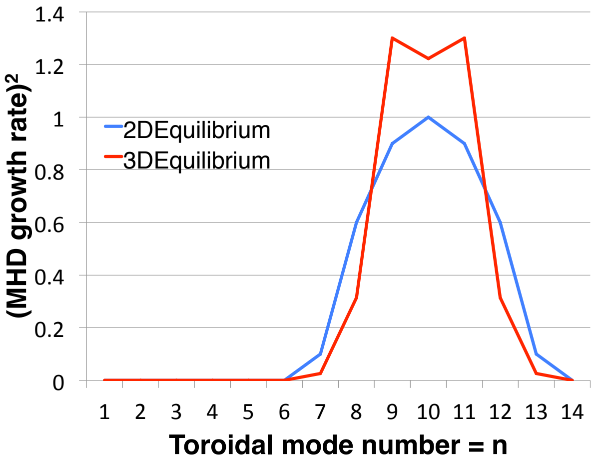

FIG. 1. The MHD stability spectrum for two equilibria are plotted as growth rate (arbitrary units) vs. toroidal mode number. The first equilibrium is axisymmetric while the second equilibrium is perturbed by a small 3-D distortion.

somewhat more peaked growth rate spectrum.

III. Application to Peeling-Ballooning Stability with RMP

As an example calculation, we consider a model problem of relevance to the issue of peeling-ballooning stability in the presence of RMPs. To describe the 3-D response to the RMPs, a physics model is required to describe the penetration of the resonant fields. With a "vacuum" response, RMPs will give rise to magnetic islands and/or regions of stochasticity. In this case, the analysis of the previous section is not directly applicable and a different approach is needed. However, if the plasma shields the resonant component from penetrating, good magnetic surfaces remain. In the limit of perfect shielding, eddy currents persist at the rational surfaces. In the following, we assume that these eddy currents provide the dominant contribution to the matrix element. Obviously a more complete characterization requires a detailed evaluation of all the terms present in the general expression given in Eq. (A3).

To estimate the contribution of the eddy currents, to the matrix element, we note $\mu_o \delta J \sim \delta B/\delta_L$ where δ_L is a characteristic layer width. With $\nabla\xi \sim k\xi$ and $\nabla B_0 \sim B_0/L_0$, we would anticipate $k\delta_L \ll 1$ and $\delta_L/L_0 \ll 1$. In this limit, the 3-D correction to force operator is dominantly

$$\delta\mathbf{F} \approx \delta\mathbf{J}\times\tilde{\mathbf{B}} = \frac{\delta J_{||}}{B}\mathbf{B}\times\tilde{\mathbf{B}}. \quad (11)$$

Further, the localized eddy currents in the perfect shielding limit are centered at their associated rational surfaces and can be written as a Fourier sum

$$\mu_o\frac{\delta J_{||}}{B} = \sum_{MN}[\lambda_{MN}e^{iM\theta-iN\zeta}+\lambda^*_{MN}e^{-iM\theta+iN\zeta}]\delta(\rho-\rho_{MN}), \quad (12)$$

where the straight field line coordinates ρ, θ and ζ are used with the poloidal flux function $\psi = \psi(\rho)$, $\psi' = d\psi/d\rho$ and $q(\rho_{MN}) = M/N$. Note in this form, this additional current is consistent with the quasi-neutrality condition as $\mathbf{B}\cdot\nabla\delta J_{||}/B \sim \sum_{MN}(M-Nq)\delta(\rho-\rho_{MN})\lambda_{MN}e^{iM\theta-iN\zeta} = 0$.

With this form for $\delta\mathbf{F}$, the matrix elements $V_{\mathbf{nk}}$ can be evaluated as a sum over the contributions associated with each rational surface. For a single N and multiple M, the matrix elements are given by

$$V_{\mathbf{nk}} = \sum_M(\lambda_{M,n-k}C^M_{\mathbf{n0k0}} + \lambda^*_{M,k-n}C^{M*}_{\mathbf{k0n0}}) \quad (13)$$

where the coefficients $C^M_{\mathbf{n0k0}}$ are defined by

$$C^M_{\mathbf{n0k0}} = -\frac{4\pi^2}{\mu_o N_{\mathbf{n0k0}}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{iM\theta+i(k-n)\zeta}\left[\frac{\psi'\vec{\xi}^*_{n0}\cdot\mathbf{B}\times\tilde{\mathbf{B}}_{k0}}{\mathbf{B}\cdot\nabla\theta}\right]_{q=\frac{M}{n-k}}, \quad (14)$$

with the normalization coefficient $N_{\mathbf{n0k0}} = (\int d^3\mathbf{x}\rho|\vec{\xi}_{\mathbf{n0}}|^2)^{1/2}(\int d^3\mathbf{x}\rho|\vec{\xi}_{\mathbf{k0}}|^2)^{1/2}$ re-introduced for clarity. If the perpendicular components of the MHD displacement vector are further decomposed using

$$\vec{\xi}_\perp = \xi^\rho\frac{\nabla\rho}{\nabla\rho\cdot\nabla\rho} + \xi^D\frac{\mathbf{B}\times\nabla\rho}{B^2}, \quad (15)$$

the coupling coefficients can be written

$$C^M_{\mathbf{n0k0}} = -\frac{4\pi^2}{\mu_o N_{\mathbf{n0k0}}}\int_0^{2\pi}\frac{d\theta}{2\pi}e^{iM\theta+i(k-n)\zeta}\times$$

$$\left\{\frac{\psi'}{\mathbf{B}\cdot\nabla\theta}[\xi^{D*}_{\mathbf{n0}}(\mathbf{B}\cdot\nabla)\xi^\rho_{\mathbf{k0}} - \xi^{\rho*}_{\mathbf{n0}}(\mathbf{B}\cdot\nabla)\xi^D_{\mathbf{k0}} + \xi^{\rho*}_{\mathbf{n0}}\xi^\rho_{\mathbf{k0}}\frac{B^2}{g^{\rho\rho}}s]\right\}_{q=\frac{M}{n-k}}. \quad (16)$$

where $s = \mathbf{B}\times\nabla\rho\cdot\nabla\times(\mathbf{B}\times\nabla\rho)/B^2g^{\rho\rho}$ is the local shear. With this model, changes to the ideal MHD stability properties are contained in the product of eddy current amplitudes and coupling coefficients computed from the properties of the eigenfunctions of the axisymmetric configuration.

This model is used in the construction of Figure 1. The spectrum and mode structure for the axisymmetric equilibrium are analytically prescribed. The matrix elements are constructed using Eq. (13) and Eq. (16) with ξ^D given by the result from high-n ballooning theory $in\xi^D_{\mathbf{n0}} = -\psi'\partial\xi^\rho_{\mathbf{n0}}/\partial\rho$ where higher order corrections in $1/n$ [10] are neglected for simplicity. These elements are subsequently used in Eq. (10) to construct the spectrum modified by the 3-D distortion.

IV. Summary

The effect of a weakly three-dimensional distortion on an otherwise axisymmetric configuration is shown to be destabilizing to ideal MHD modes. The calculation assumes that the 3-D fields are small and that shielding physics is present so that the equilibrium remains approximately topologically toroidal. A perturbation theory theory is employed that describes the coupling of different toroidal mode number harmonics in the construction of the eigenfunction. The effect of the 3-D field on the spectrum is quantified by Eq. (10) where the 3-D equilibrium enters through the matrix elements $V_{\mathbf{nk}}$ given by Eq. (6). This calculation indicates that the "small" 3-D field is destabilizing to the most unstable (or least stable) ideal MHD mode.

A model for how this formalism can be used for the case of tokamak H-mode stability in the presence of shielded RMPs is developed. For this model, the dominant component of the 3-D field is modeled as the sum of eddy current responses localized to their associated rational surfaces. This results in a calculation for the matrix elements that are products of eddy current amplitudes and a coupling coefficient computed from properties of the eigenfunctions of the axisymmetric configuration.

For more quantitative comparisons, an evaluation of the entire contribution to the matrix elements as given in Appendix A is needed. In order to compute these quantities, a measure of the 3-D distortion to the MHD equilibrium is required. A number of computational tools may be employed in this endeavor [28–30]. This allows one to compute ideal MHD stability properties of a 3-D equilibrium using knowledge of RMP amplitudes with associated plasma responses and output from an ideal MHD stability code for the associated axisymmetric equilibrium.

It is useful to compare the results presented here for the effects of small 3-D distortions on the stability properties of global MHD modes to the results of the companion work related to local MHD stability [19]. The local mode studies showed 3-D effects can produce order unity changes to marginal stability profiles, are generally destabilizing and are quite sensitive to the proximity to low-order rational surfaces. Conversely, the global mode stability arguments presented here show 3-D effects are generally destabilizing, but produce modest changes to growth rate spectra. This suggestions less sensitivity to variations in profiles for global modes relative to local stability. However, quantitative analyses of particular equilibria have yet to be carried out.

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Appendix A: Matrix Elements

To construct $\delta\mathbf{F}$, the magnetic field is considered to be the sum of an axisymmetric component and a small 3-D component

$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}, \tag{A1}$$

with the associated current satisfying $\nabla\times\delta\mathbf{B} = \mu_o\delta\mathbf{J}$. Using this, $\delta\mathbf{F}$ is given by

$$\delta\mathbf{F}(\vec{\xi}) = \delta\mathbf{J}\times[\nabla\times(\vec{\xi}\times\mathbf{B}_0)] + \mathbf{J}_0\times[\nabla\times(\vec{\xi}\times\delta\mathbf{B})]$$

$$+\frac{1}{\mu_o}\{\nabla\times[\nabla\times(\vec{\xi}\times\mathbf{B}_0)]\}\times\delta\mathbf{B}+\frac{1}{\mu_o}\{\nabla\times[\nabla\times(\vec{\xi}\times\delta\mathbf{B})]\}\times\mathbf{B}_0$$

$$+\nabla(\vec{\xi}\cdot\nabla\delta p + \gamma\delta p\nabla\cdot\vec{\xi}). \tag{A2}$$

With this form, the matrix element is given by

$$V_{\mathbf{nk}} = \int d^3\mathbf{x}(\delta\mathbf{J}\cdot\vec{\xi}^*_{\mathbf{n}0}\times\tilde{\mathbf{B}}_{\mathbf{k}0}+\delta\mathbf{B}\cdot\mathbf{D}_1+\nabla\cdot\mathbf{R}_1+C_1), \tag{A3}$$

where $\tilde{\mathbf{B}}_{\mathbf{k}0} = \nabla\times(\vec{\xi}_{\mathbf{k}0}\times\mathbf{B}_0)$ is the perturbed magnetic field associated with the ideal MHD displacement vector and

$$\mathbf{D}_1 = -\frac{1}{\mu_o}\vec{\xi}_{\mathbf{k}0}\times(\nabla\times\tilde{\mathbf{B}}^*_{\mathbf{n}0})$$

$$-\frac{1}{\mu_o}\vec{\xi}^*_{\mathbf{n}0}\times(\nabla\times\tilde{\mathbf{B}}_{\mathbf{k}0}) + \vec{\xi}_{\mathbf{k}0}\times[\nabla\times(\vec{\xi}^*_{\mathbf{n}0}\times\mathbf{J}_0)], \tag{A4}$$

$$\mathbf{R}_1 = -\frac{1}{\mu_o}(\vec{\xi}^*_{\mathbf{n}0}\times\mathbf{B}_0)\times[\nabla\times(\vec{\xi}_{\mathbf{k}0}\times\delta\mathbf{B})]$$

$$-\frac{1}{\mu_o}(\delta\mathbf{B}\times\vec{\xi}_{\mathbf{k}0})\times\tilde{\mathbf{B}}^*_{\mathbf{n}0} + (\delta\mathbf{B}\times\vec{\xi}_{\mathbf{k}0})\times(\vec{\xi}^*_{\mathbf{n}0}\times\mathbf{J}_0)$$

$$-\vec{\xi}^*_{\mathbf{n}0}\vec{\xi}_{\mathbf{k}0}\cdot\nabla\delta p - \gamma\delta p\vec{\xi}^*_{\mathbf{n}0}\nabla\cdot\vec{\xi}_{\mathbf{k}0}, \tag{A5}$$

$$C_1 = \nabla\cdot\vec{\xi}^*_{\mathbf{n}0}(\vec{\xi}_{\mathbf{k}0}\cdot\nabla\delta p + \gamma\delta p\nabla\cdot\vec{\xi}_{\mathbf{k}0}). \tag{A6}$$

An approximate estimate for the amplitudes of the matrix elements can be obtained following the calculation of Section III. The coefficients λ_{MN} representing the shielded currents at rational surfaces can be related to the radial amplitudes of the magnetic perturbations at the plasma surface using a high aspect ratio cylindrical approximation. With $r = a$ denoting the plasma surface, the shielding current amplitudes are given by

$$\lambda_{MN} \sim \frac{\delta B_{ra}}{B}\frac{2(r_{MN}/a)^{M-1}}{1-(r_{MN}/a)^{2M}}, \tag{A7}$$

for $q(r_{MN}) = M/N$. For $r_{MN}/a \sim 0.95, M \sim 10$, the scaling $\lambda_{MN} \sim 2\delta B_{ra}/B$ is obtained. To estimate the coupling coefficients C^M_{n0k0}, we use $\psi' \sim B_\theta R \sim Ba/q$,

$\mathbf{B}\cdot\nabla\theta \sim B_\theta/a \sim B/qR$, $\tilde{B} \sim B\xi/qR$, $N \sim \rho 2\pi^2 a^2 R|\xi|^2$, yielding

$$C^M_{n0k0} \sim \frac{2qR}{a}\omega_A^2, \quad \text{(A8)}$$

where $\omega_A^2 = v_A^2/q^2R^2 = B^2/(\rho\mu_o q^2 R^2)$ is the square of the Alfven frequency. To calculate $V_{\mathbf{nk}}$, we sum over contributions from each rational surface, hence $V \sim (M_{max} - M_{min} + 1)\lambda C$. For $(M_{max} - M_{min} + 1) \sim 5$ rational surfaces and $qR/a \sim 10$, we obtain the scaling

$$V_{\mathbf{nk}} \sim 200\frac{\delta B_{ra}}{B}\omega_A^2. \quad \text{(A9)}$$

For amplitudes characteristic of DIII-D [3], $\delta B_{ra}/B \sim 2\times 10^{-4}$, the matrix elements have amplitude $V \sim 0.04\omega_A^2$. Typical MHD growth rates scales crudely as $\omega_0^2 \sim (\beta R/a)\omega_A^2 \sim 0.1\omega_A^2$. This suggests that the 3-D equilibrium modification can be significant enough to modify ideal MHD stability boundaries. Careful quantitative estimates require a more complete evaluation of the matrix coefficients.

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