Coulomb collision effects on linear Landau damping

J.D. Callen*
University of Wisconsin, Madison, WI 53706-1609

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Coulomb collisions at rate ν produce slightly probabilistic rather than fully deterministic charged particle trajectories in weakly collisional plasmas. Their diffusive velocity scattering effects on the response to a wave yield an effective collision rate $\nu_{\rm eff} \gg \nu$ and a narrow dissipative boundary layer for particles with velocities near the wave phase velocity. These dissipative effects produce temporal irreversibility for times $t \gtrsim 1/\nu_{\rm eff}$ during Landau damping of a small amplitude Langmuir wave.

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I. INTRODUCTION

In a classic theoretical physics paper [1] Landau used a Laplace transform to introduce causality into the process of determining the time-asymptotic linear plasma response to a wave perturbation. This produced what is now known as Landau damping of the wave, which has been confirmed experimentally [2]. It is often thought of as a "collisionless," entropy-producing process because: collisional effects are not involved in the derivation; the damping rate is independent of the collision rate; and the wave damping seems to imply irreversibility.

However, after a wave is turned on its phase information is transferred to a perturbed distribution of the plasmas' charged particles. That this initial state information still exists has been demonstrated by showing a second wave produces a plasma echo [3, 4]. But when sufficient Coulomb collisions are introduced, echoes are damped [5, 6]. Thus, irreversibility produced during Landau damping in a plasma is due to a collisional process.

As illustrated in Fig. 1, Coulomb collisions of a charged particle with other charged particles in a plasma mainly scatter the particle's velocity vector $\mathbf{v}$. The characteristic Coulomb collision scattering rate ν is usually much smaller than the Landau damping rate γ_L. In spherical velocity space coordinates speed $v \equiv |\mathbf{v}|$, "pitch-angle" ϑ and phase angle φ, Coulomb collisions diffuse the pitch-angle through a small angle $\delta\vartheta \equiv \vartheta_0 - \vartheta \sim \sqrt{\nu_\perp \tau} \ll 1$ and cause a typically smaller diffusive spread in the speed $\delta v/v_0 \equiv (v_0 - v)/v_0 \sim \sqrt{2\,\nu_\| \tau} \ll 1$ in a time $\tau \ll 1/\nu$. These Coulomb collisional scattering effects are operative in and intrinsic to most weakly collisional plasmas.

Coulomb collision effects on the plasma response to a small amplitude wave can be illustrated by considering their effects on the phase Φ of a particle initially at $\mathbf{x}_0, \mathbf{v}_0$ at time t_0 relative to a wave with Fourier representation $\tilde{\phi}(\mathbf{x}_0, t_0) = \hat{\phi}(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{x}_0 - \omega t_0)}$ there. Here, both ω and $\mathbf{k}$ are real. Along particle trajectories $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\tau$, a short time $\tau \equiv t - t_0$ after a wave is turned on the effective phase including the weak Coulomb collisional scattering

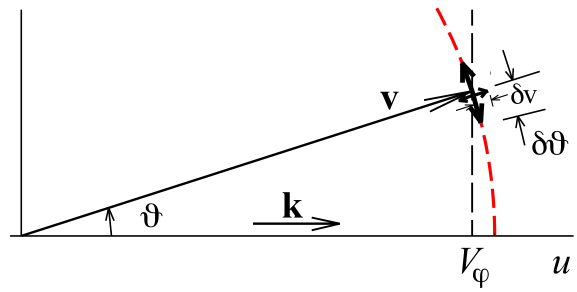

FIG. 1. Thick arrows indicate small Coulomb collisional scattering of the pitch-angle ϑ_0 about ϑ and speed v_0 about v. The wave phase speed in the $\mathbf{k}$ direction is $V_\varphi \equiv \omega/k$. Particles with $u \equiv \mathbf{v}\cdot\mathbf{k}/k = V_\varphi$ are resonant with the wave.

effects deduced from Eqs. (11)–(13) below is

$$\Phi = -(\mathbf{k}\cdot\mathbf{v}_0 - \omega)\tau + i\,\vartheta_0^2/(\nu_\perp \tau) + i\,(v - v_0)^2/(2v_0^2 \nu_\| \tau)$$
$$\simeq (\omega - ku)\tau + ku\tau\left[\frac{\vartheta_0^2 - \vartheta^2}{2} + \frac{\delta v}{u}\right] + i\frac{\delta\vartheta^2}{\nu_\perp \tau} + i\frac{\delta v^2/v_0^2}{2\,\nu_\| \tau}. \quad (1)$$

The spherical velocity-space symmetry axis is taken to be along $\mathbf{k}$ so $\mathbf{k}\cdot\mathbf{v}_0 \equiv kv_0\cos\vartheta_0 \simeq kv\,(1 + \delta v/v - \vartheta_0^2/2)$ and $ku \equiv \mathbf{k}\cdot\mathbf{v} = kv\cos\vartheta \simeq kv\,(1 - \vartheta^2/2)$.

In (1) $\omega - ku$ represents the Doppler-shifted wave frequency, $ku\tau\,[(\vartheta_0^2 - \vartheta^2)/2 + \delta v/u]$ is due to $\mathbf{k}\cdot\mathbf{v}_0 \neq ku$ and the two imaginary terms result from the dissipative pitch-angle scattering and speed diffusion effects of Coulomb collisions. The combination of all the $\delta\vartheta$ and δv terms yield an effective Coulomb collisional damping rate

$$\nu_{\rm eff} \equiv (\nu_\perp \omega/2)^{1/2} + (\nu_\| \omega^2/2)^{1/3} \gg \nu, \quad (2)$$

which causes decorrelation of particles from the wave and temporal irreversibility for $\tau \gtrsim 1/\nu_{\rm eff}$.

Collisional effects on the time-asymptotic Landau damping of a Langmuir (electron plasma) wave have been explored [6–10] using speed-diffusion-based collision operators that neglect the pitch-angle scattering effects of Coulomb collisions. Landau damping of the quasilinear wave energy has been interpreted physically [11]. Diverse applications of the Landau theory are summarized in Ref. [12]. Also, collisionless damping of a finite amplitude wave that traps particles in its sinusoidal potential has been studied [13–18].

* callen@engr.wisc.edu; http://www.cae.wisc.edu/~callen

This paper introduces a novel Green function procedure for incorporating short time scale ($\tau \ll 1/\nu$) velocity scattering effects of Coulomb collisions on electron trajectories after a small amplitude Langmuir wave is imposed. The procedure is analogous to approaches developed for a Brownian-motion-type isotropic collision operator [19] and for resonant broadening effects caused by strong plasma turbulence [20]. The approach developed in this paper provides new perspectives on the temporal evolution, resonance broadening and irreversibility involved in Landau damping in weakly collisional plasmas.

This paper is organized as follows. The next section discusses the effects of weak Coulomb collisional scattering on the plasma kinetic equation and its solutions. The following section presents a concise derivation of the Landau damping of a Langmuir wave. Next, Section IV introduces a novel Green function solution of the linearized plasma kinetic equation that includes the weak Coulomb collisional scattering effects. The penultimate section discusses how the collisional effects in the Green function solution resolve the "collisionless" singularity where $u = V_\varphi \equiv \omega/k$. The final section discuses new perspectives on Landau damping that result from this analysis.

II. COLLISION OPERATOR AND ITS EFFECTS

For times $t \lesssim 1/\nu_{\rm eff}$ the lowest order, approximate (subscript a) Fokker-Planck (F-P) Coulomb collision operator acting on a perturbed distribution $\tilde{f}_e$ of electrons with charge $q_e = -\,e$ and mass m_e is (see Appendix A)

$$\mathcal{C}_{\rm a}\{\tilde{f}_e\} \simeq \frac{\nu_\perp}{4\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\tilde{f}_e}{\partial\vartheta}\right) + \frac{\nu_\parallel v^2}{2}\,\frac{\partial^2\tilde{f}_e}{\partial v^2}. \qquad (3)$$

For Landau damping in the tail of the electron distribution [2] where $v \gg v_{Te} \equiv \sqrt{2T_e/m_e}$, $\nu_\perp \simeq 2\,(1+Z_i)\,\nu$ (electron collisions with electrons and ions of charge Z_i) and $\nu_\parallel \simeq (v_{Te}/v)^2\nu \ll \nu$. The reference collision rate is

$$\nu(v) \equiv \frac{4\pi n_e e^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2 m_e^2 v^3} = \omega_p\,\frac{\ln\Lambda}{8\sqrt{2}\,\pi\,(n_e\lambda_D^3)}\,\frac{v_{Te}^3}{v^3}. \qquad (4)$$

Here, Λ is the ratio of the collective Debye shielding length $\lambda_D \equiv (\epsilon_0 T_e/n_e e^2)^{1/2}$ (m) to the distance of closest approach during Coulomb collisions, and T_e/e (eV) and n_e (m^{-3}) are the electron temperature and density. The $\ln\Lambda$ factor (typically $\gtrsim 10$) represents the cumulative effect of all electron collisions within a Debye sphere. Also, $\omega_p \equiv (n_e e^2/m_e\epsilon_0)^{1/2}$ is the Langmuir electron plasma frequency. In typical plasmas $n_e\lambda_D^3 \gg 1$ so the collision rate ν is much smaller than ω_p and the Coulomb collision effects are weak. Note also that a very large number ($\sim n_e\lambda_D^3 \gg 1$) of statistically independent Coulomb collisions contribute to ν and hence $\mathcal{C}_a\{\tilde{f}_e\}$ during the short time $1/\omega_p$ the electron traverses a Debye sphere. The complete F-P operator $\mathcal{C}\{f_e\}$ [21] is the small momentum transfer limit [22] of the Boltzmann collision operator.

The plasma kinetic equation [22] (PKE) for the ensemble-averaged electron distribution function f_e results from equating the total time derivative of f_e to the microscopic ($|\mathbf{x}| \lesssim \lambda_D$) Coulomb collision effects on f_e:

$$\frac{df_e(\mathbf{x},\mathbf{v},t)}{dt} \equiv \frac{\partial f_e}{\partial t} + \mathbf{v}\cdot\frac{\partial f_e}{\partial \mathbf{x}} + \frac{\mathbf{F}_e}{m_e}\cdot\frac{\partial f_e}{\partial \mathbf{v}} = \mathcal{C}\{f_e\}. \quad (5)$$

Here, $\mathbf{F}_e$ is the macroscopic (i.e., $|\mathbf{x}| > \lambda_D$) electromagnetic field (Lorentz) force on electrons in the plasma.

When collisions are neglected, the resultant "collisionless" equation is called the Vlasov equation. The characteristic curves of the Vlasov first order partial differential equation are deterministic, reversible particle trajectories obtained by solving $d\mathbf{x}/dt = \mathbf{v}$, $d\mathbf{v}/dt = \mathbf{F}_e/m_e$ for $\mathbf{x}$, $\mathbf{v}$.

The full PKE includes the Coulomb collision operator which has second derivatives in velocity space. Thus, the PKE is a parabolic second order partial differential equation (i.e., a diffusion equation); its particle trajectories are probabilistic and hence temporally irreversible. When the Coulomb collision damping rate is small (i.e., $|\mathcal{C}\{f_e\}| \ll |df_e/dt|$ for most $\mathbf{x},\mathbf{v}$), (5) becomes a singular differential equation. Collisions smooth out responses in velocity regions $\delta\vartheta \sim \sqrt{\nu_\perp\tau} \ll 1$ and $\delta v/v \sim \sqrt{2\,\nu_\parallel\tau} \ll 1$ and create dissipative singular layers where df_e/dt is smaller than $\nu_{\rm eff} f_e \sim [\nu_\perp/2(\delta\vartheta)^2 + \nu_\parallel v^2/(\delta v)^2] f_e \gg \nu f_e$.

III. LANDAU DAMPING

Landau damping will be explored in an infinite, homogeneous, unmagnetized plasma that has only an electrostatic electric field force $\mathbf{F}_e = -\,q_e\boldsymbol{\nabla}\phi$. Assuming the equilibrium has no macroscopic electric field, the equilibrium PKE is $0 = \mathcal{C}\{f_{e0}\}$. The general solution of this equation is the isotropic Maxwellian distribution: $f_{e0} = f_{\rm Me}(v) \equiv [n_{e0}/(\pi^{3/2}v_{Te}^3)]\,e^{-v^2/v_{Te}^2}$.

Introducing a small electric field perturbation $\tilde{\mathbf{E}} \equiv -\boldsymbol{\nabla}\tilde{\phi}$ in (5), $f_e \to f_{\rm Me} + \tilde{f}_e$, which yields a linearized perturbed PKE for the perturbed distribution $\tilde{f}_e(\mathbf{x},\mathbf{v},t)$:

$$\frac{\partial\tilde{f}_e}{\partial t} + \mathbf{v}\cdot\frac{\partial\tilde{f}_e}{\partial\mathbf{x}} - \mathcal{C}_{\rm a}\{\tilde{f}_e\} = -\,\frac{q_e}{T_e}(\mathbf{v}\cdot\boldsymbol{\nabla}\tilde{\phi})f_{\rm Me}. \quad (6)$$

Neglecting collision effects and assuming $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ Fourier perturbations for $\tilde{\phi}$ and $\tilde{f}_e$, this equation yields

$$\hat{f}_e = \frac{(q_e/T_e)\,(\mathbf{k}\cdot\mathbf{v})\,\hat{\phi}\,f_{\rm Me}}{\omega - \mathbf{k}\cdot\mathbf{v}} = \frac{e\hat{\phi}}{T_e}f_{\rm Me}\frac{u}{u - V_\varphi}, \qquad (7)$$

whose "resonant denominator" $1/(u - V_\varphi)$ indicates a singular response for particles that have speeds $u \equiv \mathbf{k}\cdot\mathbf{v}/k$ along $\mathbf{k}$ equal to the wave phase speed $V_\varphi \equiv \omega/k$.

Since the solution in (7) is real, Vlasov [23] concluded there is no damping of wave-like perturbations in low collisionality plasmas. Landau [1] also neglected collision effects but used a Laplace transform for the time domain to employ causality to define the singularity in (7) by deforming the inverse Laplace transform contour to always be below this singularity. Using $V_\varphi \to V_\varphi + i\gamma/k$, for small γ this formalism yields the Plemelj formula

$$\lim_{\gamma\to 0} \frac{u}{u-(V_\varphi+i\gamma/k)} \doteq \mathcal{P}\left\{\frac{u}{u-V_\varphi}\right\}+i\pi\, u\,\delta[u-V_\varphi]. \quad (8)$$

In this Landau prescription for resolving the singularity at $u = V_\varphi$, $\mathcal{P}\{F/x\} \equiv \lim_{\epsilon\to 0}\{\int_{-\infty}^{-\epsilon} F dx/x + \int_\epsilon^\infty F dx/x\}$ is the (real) principal value operator and $\delta[x]$ is the Dirac delta function.

The perturbed electron density induced by $\tilde{\phi}$ is obtained by integrating $\hat{f}_e$ in (7) over all velocity space using (8). For $V_\varphi \gg v_{Te}$ this yields [22, 24] $\tilde{n}_e \equiv \int d^3v\, \tilde{f}_e = -n_{e0}\,(e\tilde{\phi}/T_e)\,[v_{Te}^2/2V_\varphi^2 + 3v_{Te}^4/4V_\varphi^4 + \cdots + i\sqrt{\pi}(V_\varphi/v_{Te})e^{-V_\varphi^2/v_{Te}^2}]$. The $k\lambda_D \ll 1$ dispersion relation for Langmuir waves is obtained [24] by substituting this $\tilde{n}_e$ into the perturbed Poisson equation $-\nabla^2\tilde{\phi} = \tilde{\rho}_q/\epsilon_0 \simeq -\tilde{n}_e e/\epsilon_0$ in which the small ion contribution has been neglected. Solving this dispersion relation for the complex frequency $\omega \equiv \omega_R + i\,\gamma$ yields the real frequency $\omega_R \simeq \omega_p\,[1+(3/2)\,k^2\lambda_D^2+\cdots]$ and Landau damping rate

$$\gamma_L \simeq -\frac{\omega_p}{(k\lambda_D)^3}\left(\frac{\pi}{8}\right)^{1/2} e^{-1/(2\,k^2\lambda_D^2)-3/2}. \quad (9)$$

The imaginary delta function term in (8) produces γ_L, which is usually smaller than ω_p but much larger than ν. For example, for $k\lambda_D \simeq 0.3$ and $V_\varphi/v_{Te} \simeq 2.7$ in the $n_e\lambda_D^3 \sim 10^4$ plasmas in Ref. [2] where $\nu(v_{Te})/\omega_p \sim 10^{-4}$, $\gamma_L/\omega_p \simeq 2\times 10^{-2}$ and $\nu_\parallel/\nu_\perp \sim 1/30$.

IV. GREEN FUNCTION SOLUTION

In spherical velocity-space coordinates $u \equiv \mathbf{v}\cdot\mathbf{k}/k = v\cos\vartheta$. Thus, if the collision operator in (3) operates on $\hat{f}_e$ in (7), it yields $\mathcal{C}_{\rm a}\{\hat{f}_e\} \propto [\nu_\perp \sin^2\vartheta + 2\,\nu_\parallel]/(u-V_\varphi)^3$, which is even more singular than $\hat{f}_e$. This paper introduces a Green function procedure for solving (6) that collisionally resolves the singular region $u \simeq V_\varphi$.

Since for short times $\delta\vartheta \sim \sqrt{\nu_\perp\tau} \ll 1$, the collision operator in (3) can be simplified using $\sin\vartheta \simeq \vartheta$ for $\vartheta \ll 1$. Then, the perturbed PKE in (6) becomes

$$\frac{\partial \tilde{f}_e}{\partial t}+\mathbf{v}\cdot\frac{\partial \tilde{f}_e}{\partial \mathbf{x}}-\frac{\nu_\perp}{4\vartheta}\frac{\partial}{\partial\vartheta}\left(\vartheta\frac{\partial \tilde{f}_e}{\partial\vartheta}\right)-\frac{\nu_\parallel v^2}{2}\frac{\partial^2 \tilde{f}_e}{\partial v^2} = \frac{e\mathbf{v}\cdot\boldsymbol{\nabla}\tilde{\phi}}{T_e} f_{\rm Me}. \quad (10)$$

This result is in the form $\mathcal{L}\{\tilde{f}_e\} = (e/T_e)\,(\mathbf{v}\cdot\boldsymbol{\nabla}\tilde{\phi})f_{\rm Me}$ in which $\mathcal{L}$ is a linear partial differential operator. Its solution can be obtained in terms of a Green function $G(\mathbf{x},\vartheta,v|\mathbf{x}_0,\vartheta_0,v_0;t-t_0)$ that solves the equation $\mathcal{L}\{G\} = \delta[\mathbf{x}-\mathbf{x}_0]\,\delta[(\vartheta-\vartheta_0)^2/2]\,\delta[v-v_0]\,\delta[t-t_0]$. This Green function represents the response at $\mathbf{x},\vartheta,v,t$ to a delta function source at $\mathbf{x}_0,\vartheta_0,v_0,t_0$. Assuming that $\tilde{f}_e = 0$ initially, the Green function solution of (10) is

$$\tilde{f}_e = \int_0^t dt_0 \int d^3x_0 \int_0^\infty 2\pi v_0^2\, dv_0 \int_0^\pi \sin\vartheta_0\, d\vartheta_0 \times G\,\frac{e}{T_e}\left[(\mathbf{v}\cdot\boldsymbol{\nabla}\tilde{\phi})f_{\rm Me}\right]_{\mathbf{x}_0,\vartheta_0,v_0,t_0}. \quad (11)$$

The characteristic curves of the first derivative (Vlasov) operators in (10) are obtained by solving $d\mathbf{x}/dt = \mathbf{v}$ and $d\mathbf{v}/dt = \mathbf{0}$. Using initial conditions $\mathbf{x} = \mathbf{x}_0$ and $\mathbf{v} = \mathbf{v}_0$ (i.e., $v = v_0$ and $\vartheta = \vartheta_0$) at the initial time t_0, these first order ordinary differential equations yield the reversible, deterministic particle trajectory $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\tau$ in which $\tau \equiv t - t_0 \geq 0$. When collisional effects are weak the effects of the second derivative collision operator in (10) are separable from these first order characteristics. For $\nu\tau \ll 1$ the Coulomb-collision-induced characteristics are $e^{-(\vartheta-\vartheta_0)^2/(\nu_\perp\tau)}/(\nu_\perp\tau/2)$ for pitch-angle scattering and $e^{-(v-v_0)^2/(2v_0^2\nu_\parallel\tau)}/(v_0\sqrt{2\pi\nu_\parallel\tau}\,)$ for speed diffusion.

By construction, the complete Green function for the differential operator on the left hand side of (10) is thus

$$G = \delta[\mathbf{x}-\mathbf{x}_0-\mathbf{v}_0\tau]\,\frac{e^{-(v-v_0)^2/(2v_0^2\nu_\parallel\tau)}}{2\pi v_0^3\sqrt{2\pi\nu_\parallel\tau}}\,\frac{e^{-(\vartheta-\vartheta_0)^2/(\nu_\perp\tau)}}{\nu_\perp\tau/2}H(\tau), \quad (12)$$

in which $H(\tau) = H(t-t_0)$ is the Heaviside step function. Since $dH(\tau)/dt = \delta[\tau]$ and using the facts that $\lim_{\tau\to 0^+} e^{-(v_0-v)^2/(2v_0^2\nu_\parallel\tau)}/(v_0\sqrt{2\pi\nu_\parallel\tau}) \doteq \delta[v-v_0]$ and $\lim_{\tau\to 0^+} e^{-(\vartheta-\vartheta_0)^2/(\nu_\perp\tau)}/(\nu_\perp\tau/2) \doteq \delta[(\vartheta-\vartheta_0)^2/2]$, operating on (12) with the linear operator $\mathcal{L}$ shows G satisfies the defining equation for the Green function for $\vartheta_0 \ll 1$.

Equation (12) is the "propagator" for charged particle trajectories. It includes both the reversible, deterministic motion $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\tau$ and probabilistic, irreversible diffusion of the velocity-space speed $\delta v \sim v_0\sqrt{2\,\nu_\parallel\tau}$ and pitch-angle $\delta\vartheta \sim \sqrt{\nu_\perp\tau}$ induced by Coulomb collisions.

After substituting the Green function solution in (12) into (11), the d^3x_0 integration can be performed using the deterministic particle trajectory delta function to replace $\mathbf{x}_0$ with $\mathbf{x}-\mathbf{v}_0\tau$. Then, using $t_0 = t-\tau$ and omitting the common factor $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega)t}$, the solution in (11) yields

$$\hat{f}_e = \frac{e\hat{\phi}}{T_e} f_{\rm Me}\int_0^t d\tau \int_0^\infty \frac{dv_0}{v_0\sqrt{2\pi\nu_\parallel\tau}}\int_0^\pi \frac{\sin\vartheta_0 d\vartheta_0}{\nu_\perp\tau/2}\, i(\mathbf{k}\cdot\mathbf{v}_0)\, e^{i\Phi}, \quad (13)$$

in which Φ is the wave-particle phase defined in (1). When $e^{i\Phi}$ is integrated over ϑ_0 and ϑ, these angles are limited to $\max\{\vartheta_0,\vartheta\} \sim \sqrt{\nu_\perp\tau} \ll 1$. Also, the speed diffusion is small: $\max\{\delta v\}/v_0 \sim \sqrt{2\,\nu_\parallel\tau} \ll 1$.

Neglecting $\mathcal{O}\{\vartheta^2,\vartheta_0^2,\delta v/v_0\}$ corrections to $\mathbf{k}\cdot\mathbf{v}_0$ in (13) and setting it to ku, the solution in (13) becomes

$$\hat{f}_e = \frac{e\hat{\phi}}{T_e}\, f_{\rm Me}\, I_\nu(kut,\Delta_u,\epsilon_\parallel,\epsilon_\perp,\vartheta), \quad \text{in which} \quad (14)$$

$$I_\nu \equiv \int_0^{kut} dz\, i\, e^{-i\Delta_u z}\int_{-\infty}^\infty \frac{d\,\delta v/v_0}{\sqrt{4\pi\epsilon_\parallel z}} e^{-(\delta v/v_0)^2/(4\epsilon_\parallel z)-iz\,\delta v/v_0} \times \int_0^\infty \frac{\vartheta_0 d\vartheta_0}{\epsilon_\perp z} e^{-(\vartheta_0-\vartheta)^2/(2\,\epsilon_\perp z)+iz(\vartheta_0^2-\vartheta^2)/2}. \quad (15)$$

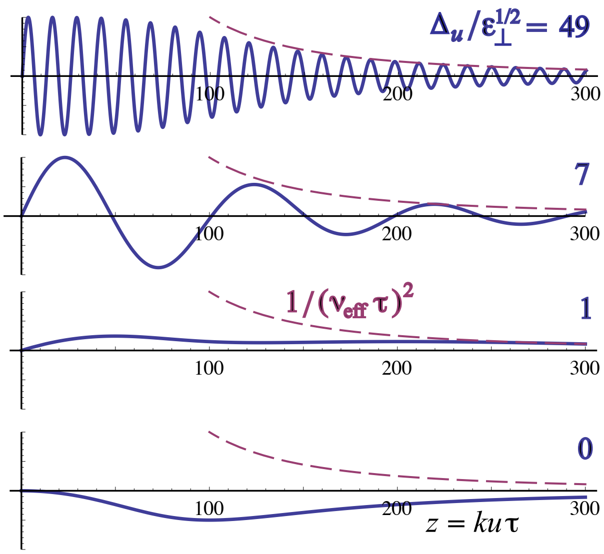

FIG. 2. Real part of $\epsilon_\parallel = 0$ integrand $i\,e^{-i\,\Delta_u z}/(1 - i\,\epsilon_\perp z^2)$ in $I_{\nu 0}$ integral which represents the in-phase distribution function response to a wave as a function of dimensionless time $z \equiv ku\tau$ after wave turn on for various Δ_u. Here $\epsilon_\perp = 10^{-4}$.

Four dimensionless parameters have been defined in the Coulomb-collision-influenced time-history integral I_ν:

$$z \equiv ku\tau, \;\; 2\pi \times \text{number of wavelengths traversed,} \quad (16)$$

$$\Delta_u \equiv \frac{u - \omega/k}{u} = 1 - \frac{V_\varphi}{u}, \;\; \text{relative speed,} \quad (17)$$

$$\epsilon_\parallel \equiv \frac{\nu_\parallel}{2\,ku} \simeq \frac{\nu_\parallel}{2\,\omega} \ll 1, \;\; \text{speed diffusion rate and} \quad (18)$$

$$\epsilon_\perp \equiv \frac{\nu_\perp}{2\,ku} \simeq \frac{\nu_\perp}{2\,\omega} \ll 1, \;\; \text{pitch-angle scattering rate.} \quad (19)$$

The ϑ_0 integral in the I_ν in (15) can be performed. This result is shown in Appendix B. For $\vartheta = 0$ it yields

$$I_{\nu 0} \;\equiv\; I_\nu(\vartheta = 0) \;=\; \int_0^{kut} dz\, \frac{i\,e^{-i\,\Delta_u z \,-\, \epsilon_\parallel z^3}}{1 - i\,\epsilon_\perp z^2}. \qquad (20)$$

The temporal evolution of the integrand of $I_{\nu 0}$ is illustrated for $\epsilon_\parallel = 0$ in Fig. 2 for various Δ_u. This pitch-angle scattering response is oscillatory for $|\Delta_u| \gg \epsilon_\perp^{1/2}$ and always in phase with the Doppler-shifted wave for $z = ku\tau \simeq \omega\tau \ll \epsilon_\perp^{-1/2}$ (= 100 here). As indicated, since I_ν is strongly peaked near $|\Delta_u| \simeq 0$ (see Figs. 4–6), ku will be approximated by ω. For $\epsilon_\parallel = 0$, the effective collision frequency in (2) is $\nu_{\rm eff} = (\nu_\perp \omega/2)^{1/2} = \omega\,\epsilon_\perp^{1/2}$. Thus, as Fig. 2 shows, the pitch-angle response is damped for all Δ_u as $1/\epsilon_\perp z^2 = 1/(\nu_{\rm eff}\,\tau)^2$ for $\tau > 1/\nu_{\rm eff}$.

The analogous temporal evolution of the integrand of $I_{\nu 0}$ for $\epsilon_\perp = 0$ is illustrated in Fig. 3 for various Δ_u. This speed diffusion response is oscillatory for $|\Delta_u| \gg \epsilon_\parallel^{1/3}$ and always in phase with the Doppler-shifted wave for $z = ku\tau \simeq \omega\tau \ll \epsilon_\parallel^{-1/3}$ ($\simeq$ 67 here). For $\epsilon_\perp = 0$ the

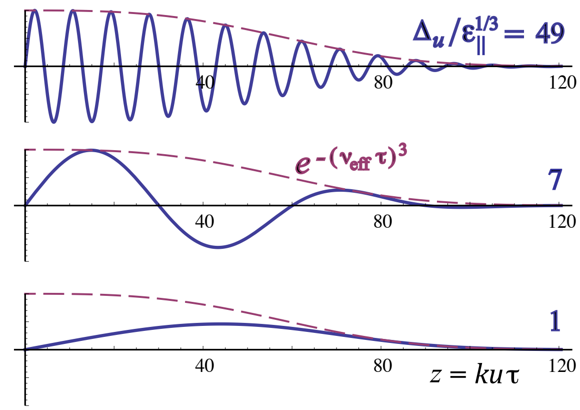

FIG. 3. Real part of the $\epsilon_\perp = 0$ integrand $i\,e^{-i\,\Delta_u z - \epsilon_\parallel z^3}$ in the $I_{\nu 0}$ integral which represents the in-phase distribution function response to a wave as a function of dimensionless time $z \equiv ku\tau$ after wave turn on for various Δ_u. Here $\epsilon_\parallel = 10^{-4}/30$.

effective collision frequency is $\nu_{\rm eff} = (\nu_\parallel \omega^2/2)^{1/3} = \omega\,\epsilon_\parallel^{1/3}$. Thus, as Fig. 3 shows, the speed diffusion response is damped for all Δ_u as [6] $e^{-\epsilon_\parallel z^3} = e^{-\nu_{\rm eff}^3 \tau^3}$ for $\tau > 1/\nu_{\rm eff}$.

The Padé approximate of these two temporally irreversible Coulomb collisional scattering effects yields the overall effective collision rate $\nu_{\rm eff}$ in (2). The $\nu_\parallel^{1/3}$ contribution to $\nu_{\rm eff}$ is often slightly larger even though on the tail of the electron distribution $\nu_\parallel \ll \nu_\perp$. However, the physically different pitch-angle scattering at rate $\nu_\perp$ and speed diffusion $\nu_\parallel$ Coulomb collisional effects act simultaneously. Thus, in general both should be included in comprehensive analyses of the temporal evolution and irreversibility in linear Landau damping.

V. COLLISIONAL RESOLUTION OF SINGULARITY AT $u = V_\varphi$

Coulomb collision effects can be neglected for short times by taking $\epsilon_\parallel z \ll 1$ and $\epsilon_\perp z \ll 1$ limits inside the I_ν in (15). Using $\lim_{\epsilon_\parallel z \ll 1} e^{-\delta v^2/(4\epsilon_\parallel z v_0^2)}/\sqrt{4\epsilon_\parallel z} \doteq \delta[\delta v/v_0]$ and $\lim_{\epsilon_\perp z \ll 1} e^{-(\vartheta_0 - \vartheta)^2/(2\epsilon_\perp z)}/(\epsilon_\perp z) \doteq \delta[\vartheta_0 - \vartheta]/\vartheta_0$, (15) yields a temporally reversible result for times $t \ll 1/\nu_{\rm eff}$:

$$\lim_{\epsilon_\parallel z, \epsilon_\perp z \ll 1} I_\nu = \frac{[1 - \cos(kut\Delta_u)] + i \sin(kut\Delta_u)]}{\Delta_u}. \quad (21)$$

Figure 4 shows the behavior of this collisionless result.

Averaging (21) in time t over the effective sinusoidal particle period $2\pi/ku$ yields a type of "phase mixed" $\overline{I}_\nu$. As indicated in Fig. 4, its real part yields $1/\Delta_u$ except near $\Delta_u = 0$ where it vanishes. Its imaginary part is largest for $|\Delta_u| \lesssim 1/kut$. Since $\sin(kut\,\Delta_u)/\Delta_u$ is a correlation function, its $kut \gg 1$ limit yields a delta function: $\lim_{kut\to\infty} \sin(kut\,\Delta_u)/\Delta_u \doteq \pi\,\delta[\Delta_u]$. Thus, the time-asymptotic limit of this phase mixed $\overline{I}_\nu$ yields $\lim_{kut\to\infty} \overline{I}_\nu(\epsilon_\parallel, \epsilon_\perp \to 0) \doteq \mathcal{P}\{1/\Delta_u\} + i\pi\,\delta[\Delta_u]$, which

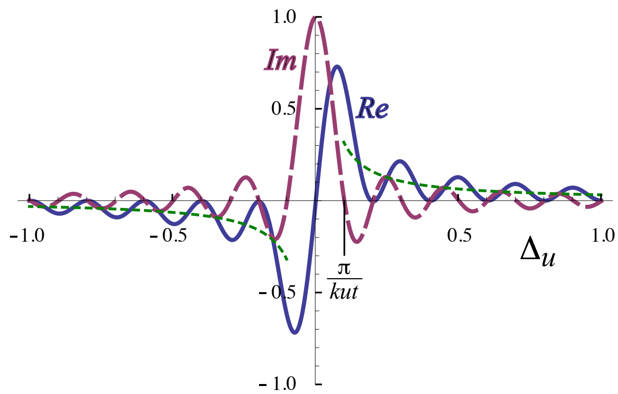

FIG. 4. Real and imaginary parts of the collisionless limit ($\epsilon_\parallel, \epsilon_\perp \to 0$) of the normalized integral I_ν/kut as a function of Δ_u for $kut = 10\pi$. The shorter dashed lines show the effects of phase mixing the real part: $\mathcal{R}e\{\bar{I}_\nu(\epsilon_\parallel, \epsilon_\perp \to 0)\} \doteq \mathcal{P}\{1/\Delta_u\}$.

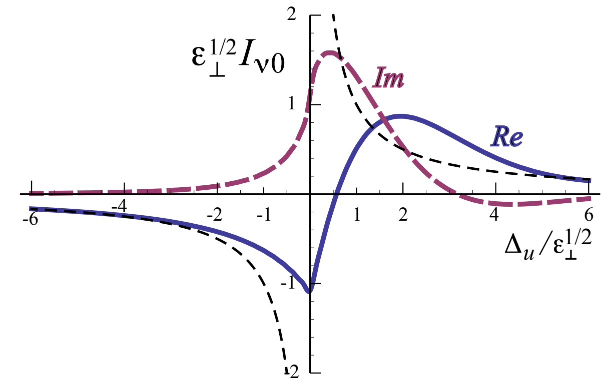

FIG. 5. Real and imaginary parts of the $\epsilon_\parallel = 0$ integral $I_{\nu 0}$ as a function of $\Delta_u/\epsilon_\perp^{1/2} \equiv (u - V_\varphi)/u\,\epsilon_\perp^{1/2}$ for $\epsilon_\perp \equiv \nu_\perp/(2\,ku) = 10^{-4}$ and $kut \gg \omega/\nu_{\rm eff} = \epsilon_\perp^{-1/2} = 100$. Thin dashed lines show the collisionless time-asymptotic behavior $\mathcal{R}e\{\bar{I}_{\nu 0}(\epsilon_\parallel, \epsilon_\perp \to 0)\} = \mathcal{P}\{1/\Delta_u\}$.

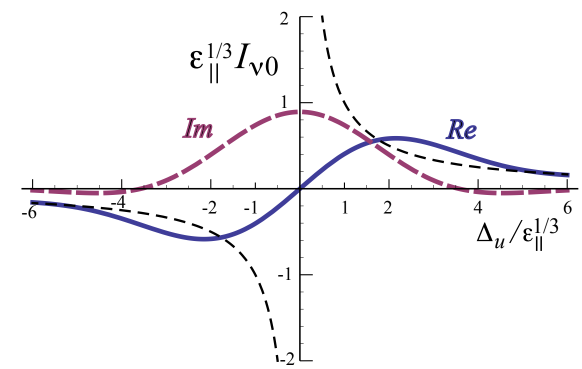

FIG. 6. Real and imaginary parts of the $\epsilon_\perp = 0$ integral $I_{\nu 0}$ as a function of $\Delta_u/\epsilon_\parallel^{1/3} \equiv (u - V_\varphi)/u\,\epsilon_\parallel^{1/3}$ for $\epsilon_\parallel \equiv \nu_\parallel/(2\,ku) = 10^{-4}/30$ and $kut \gg \omega/\nu_{\rm eff} = \epsilon_\parallel^{-1/3} \simeq 67$. Thin dashed lines show the collisionless time-asymptotic behavior $\mathcal{R}e\{\bar{I}_{\nu 0}(\epsilon_\parallel, \epsilon_\perp \to 0)\} = \mathcal{P}\{1/\Delta_u\}$.

is Landau's Laplace-transform-based prescription in (8) that is used in collisionless (Vlasov) plasma models.

However, in the presence of weak Coulomb collisional scattering effects the short time criterion $t \ll 1/\nu_{\rm eff}$ for obtaining (21) and the $kut \to \infty$ limit are incompatible. As will be shown next, the phase mixing approximation and $t \to \infty$ requirements are not needed to obtain (8) for most plasmas which by definition include weak, non-negligible, intrinsic Coulomb collisional scattering effects.

When Coulomb collision effects are included in I_ν, in general the integrals indicated in (15) have to be evaluated numerically. The behavior of the $\epsilon_\parallel = 0$, $\vartheta = 0$ time history integral $I_{\nu 0}$ in the dissipative resonance region $|(u - V_\varphi)/u| \sim \epsilon_\perp^{1/2}$ is shown in Fig. 5. Its key properties for $t \gg 1/\nu_{\rm eff} = (\nu_\perp \omega/2)^{-1/2} = 1/(\omega\epsilon_\perp^{1/2})$, which show that for $\epsilon_\perp^{1/2} \ll 1$ its real part is $\mathcal{P}\{1/\Delta_u\}$ and its imaginary part is $i\pi\delta[\Delta_u]$, are

$$\lim_{\epsilon_\perp^{1/2} \ll 1} \max\{I_{\nu 0}\} \propto \epsilon_\perp^{-1/2}, \tag{22}$$

$$\lim_{|\Delta_u|/\epsilon_\perp^{1/2} \gg 1} \{I_{\nu 0}\} \simeq \frac{1}{\Delta_u} + i\,\mathcal{O}\left\{e^{-\sqrt{2}|\Delta_u|/\epsilon_\perp^{1/2}}\right\}, \tag{23}$$

$$\int_{-\infty}^{\infty} d\left(\frac{u - V_\varphi}{u}\right) \lim_{\epsilon_\perp^{1/2} \ll 1} \{I_{\nu 0}\} \simeq 0 + i\,\pi. \tag{24}$$

The last result is obtained by numerical integration and accurate to $< 0.01\%$ for $\epsilon_\perp < 0.1$. The speed u where $\mathcal{I}m\{I_{\nu 0}\}$ peaks is larger than V_φ because Coulomb-collision-induced pitch-angle scattering causes the effective $u \simeq v\,(1 - \vartheta^2/2)$ to be smaller than v (see Fig. 1).

The analogous behavior of the $\epsilon_\perp = 0$, $\vartheta = 0$ time history integral $I_{\nu 0}$ in the dissipative resonance region $|(u - V_\varphi)/u| \sim \epsilon_\parallel^{1/3}$ is shown in Fig. 6. Its key properties for $t \gg 1/\nu_{\rm eff} = (\nu_\parallel \omega^2/2)^{-1/3} = 1/(\omega\epsilon_\parallel^{1/3})$, which show that for $\epsilon_\parallel^{1/3} \ll 1$ its real part is $\mathcal{P}\{1/\Delta_u\}$ and its imaginary part is $i\pi\delta[\Delta_u]$, are

$$\lim_{\epsilon_\parallel^{1/3} \ll 1} \max\{I_{\nu 0}\} \propto \epsilon_\parallel^{-1/3}, \tag{25}$$

$$\lim_{|\Delta_u|/\epsilon_\parallel^{1/3} \gg 1} \{I_{\nu 0}\} \simeq \frac{1}{\Delta_u} + i\,\mathcal{O}\left\{e^{-0.27|\Delta_u|^{3/2}/\epsilon_\parallel^{1/2}}\right\}, \tag{26}$$

$$\int_{-\infty}^{\infty} d\left(\frac{u - V_\varphi}{u}\right) \lim_{\epsilon_\parallel^{1/3} \ll 1} \{I_{\nu 0}\} \simeq 0 + i\,\pi. \tag{27}$$

The last result is obtained by numerical integration and accurate to $< 0.01\%$ for $\epsilon_\parallel < 0.1$. The speed u where $\mathcal{I}m\{I_{\nu 0}\}$ peaks is at $u = 0$ because at $\vartheta = 0$ the collision-induced speed diffusion is in the u direction (see Fig. 1). With collisional speed reduction the $\mathcal{I}m\{I_{\nu 0}\}$ peak moves very slightly to $\Delta_u/\epsilon_\parallel^{1/3} \sim -\epsilon_\parallel^{1/3}$ (see Appendix A).

Thus, for times $t \gtrsim 1/\nu_{\rm eff}$ Coulomb collisions produce two important effects on the collisionless I_ν shown in Fig. 4. First, far away from the resonance (i.e., for $|\Delta_u| \gg \pi/kut$) they smooth its real part to the phase mixed result $\mathcal{R}e\{\overline{I}_\nu\} \simeq 1/\Delta_u$ and cause its imaginary part to become exponentially small. Second, they limit progression of the singular layer in Fig. 4 to an ever narrower region as kut increases beyond $\omega/\nu_{\rm eff}$; they instead produce a resonance-broadened dissipative region where $|\Delta_u| \lesssim \nu_{\rm eff}/\omega$. As shown in (22)–(27), $\mathcal{I}m\{I_\nu\}$ has delta-function-type properties for small $\epsilon_\perp$ and $\epsilon_\|$: $\max\{I_\nu\} \sim \omega/\nu_{\rm eff} \gg 1$ and $\int_{-\infty}^{\infty} d\Delta_u \, I_\nu = i\pi$. Thus, for weakly collisional plasmas where $\epsilon_\perp \ll 1$ and $\epsilon_\| \ll 1$

$$\lim_{t \gg 1/\nu_{\rm eff}} I_\nu \doteq \mathcal{P}\{1/\Delta_u\} + i\pi\,\delta[\Delta_u]. \tag{28}$$

This collision-based result is Landau's prescription in (8).

VI. DISCUSSION AND CONCLUSIONS

The analysis in this paper shows that when Coulomb collisional scattering effects are included, the deterministic, reversible particle motion propagator $\delta[\mathbf{x} - \mathbf{x}_0 - \mathbf{v}_0\tau]$ is replaced by the probabilistic, irreversible Green function in (12). This causes the singular factor $u/(u - V_\varphi)$ in (7) to be replaced by the integral I_ν defined in (15), as indicated in (14). Since the $t \gg 1/\nu_{\rm eff}$ limit of I_ν yields the Landau prescription in (8), the Landau damping rate γ_L in (9) is also obtained with this Coulomb-collision-based Green function approach. Thus, a Laplace transform procedure is neither needed nor appropriate when the intrinsic, weak Coulomb collisional scattering effects that are intrinsic to most plasmas are included.

These results do not change the time-asymptotic Landau singularity resolving prescription in (8) or Langmuir wave damping rate. However, the Green function analysis presented here facilitates exploration of the plasma response to a wave on various time scales. The Coulomb-collision-influenced time-history integral I_ν includes: 1) the short-time-scale ($t \ll 1/\nu_{\rm eff}$) temporally reversible collisionless result in (21), 2) evolution into an irreversible, dissipative, resonance-broadened response for $t \gtrsim 1/\nu_{\rm eff}$ (e.g., as in damping of echoes [5, 6]) and 3) the long but finite time-scale ($t \gg 1/\nu_{\rm eff}$) Coulomb collisional justification in (28) of the Landau prescription in (8) for resolving the $u = V_\varphi$ singularity.

The collisionless limit of the time-history integral given in (21) has properties that are superficially similar to the Landau prescription in (8), especially after this collisionless I_ν is phase mixed in time. Namely, as shown in Fig. 4, its real and imaginary parts are strongly peaked in the $|\Delta_u| = |(u - V_\varphi)/u| \lesssim \pi/kut$ resonance region and its real part decays approximately as $1/\Delta_u$. However, there is a very important difference: the collisionless I_ν in (21) is temporally reversible whereas the imaginary part of (8) indicates irreversibility. A temporally irreversible response is obtained from (21) only in the time-asymptotic limit $kut \to \infty$. This is consistent with Landau's use of the Laplace transform, which introduces causality.

In the presence of weak Coulomb collisional scattering effects the short time criterion $t \ll 1/\nu_{\rm eff}$ for obtaining (21) and the time-asymptotic limit $kut \to \infty$ are incompatible. This paper has demonstrated that the probabilistic Coulomb collisional scattering effects determine the minimum width of the wave-particle resonance for $t \gtrsim 1/\nu_{\rm eff}$ and that the plasma response evolves into a temporally irreversible, dissipative state for $t \gg 1/\nu_{\rm eff}$. These results are the linear theory predictions for the long time scale plasma response to a small amplitude wave that should occur in most physically relevant plasmas which are intrinsically weakly collisional.

Does Landau damping increase entropy? Addressing this question requires a quasilinear analysis that is beyond the scope of this paper. However, Figs. 2 and 3 show the instantaneous $\tilde{f}_e$ response is in phase with the applied potential $\tilde{\phi}$ for all Δ_u up until the Coulomb-collision-induced effective collision time $1/\nu_{\rm eff}$, after which it is damped. This suggests entropy changes little for $t \ll 1/\nu_{\rm eff}$ but then increases for $t \gtrsim 1/\nu_{\rm eff}$. The temporal irreversibility is caused by the Coulomb collisional scattering effects that resolve the wave-particle singularity in the resonance region $|(u - V_\varphi)/u| \lesssim \nu_{\rm eff}/\omega$, as indicated by the localized imaginary part of I_ν in Figs. 5 and 6.

A wave of finite amplitude traps nearly resonant electrons in its sinusoidal potential. The oscillation frequency for such trapped electrons is [13–17] $\omega_{\rm trap} \equiv k\,(e\hat{\phi}/m_e)^{1/2} = (kv_{Te}/\sqrt{2}\,)\,(e\hat{\phi}/T_e)^{1/2}$. Trapping dominates when resonant electrons complete a bounce period. Thus, Coulomb collision scattering effects are dominant for times t up to the minimum of $1/\nu_{\rm eff}$ or $2\pi/\omega_{\rm trap}$. They are also critical for producing temporal irreversibility for finite times $t \gg 1/\nu_{\rm eff}$ in weakly collisional plasmas when particle trapping becomes important or even dominant.

The Landau analysis [1] of wave damping is a linear theory. However, it has been demonstrated recently [16–18] that when the collisionless Vlasov equation is used, obtaining temporal irreversibility in Landau damping on long time scales requires consideration of nonlinear effects. In particular, third order (echo-type responses) and higher order terms produce heteroclinic (temporally irreversible) solutions in which dynamical chaos ensues when a KAM-type condition for localized solutions is violated on very long but finite time scales. When the probabilistic Coulomb collisional scattering effects discussed in this paper are large enough they damp echoes [5, 6]. It remains to be determined how these collisional effects interact with and potentially modify the nonlinear effects discussed in Refs. [16–18] for most physically relevant plasmas which are intrinsically weakly collisional.

The novel Green function procedure developed in this paper for exploring Coulomb collisional scattering effects on the temporal evolution of the plasma response to a wave and the wave-particle resonance in weakly collisional plasmas should be useful for other plasma applications. It could be particularly helpful in determining the effective collision frequency that is relevant for resonance broadening and temporal irreversibility in plasmas.

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Appendix A: Approximate Fokker-Planck Coulomb Collision Operator

The "test particle" (subscript t) Fokker-Planck (F-P) Coulomb collision operator [21] on a distribution function perturbation $\tilde{f}_\alpha$ of a charged particle species α colliding with background particles β (including species α particles) that have Maxwellian velocity distributions is [25]

$$\mathcal{C}_{\rm t}\{\tilde{f}_\alpha\} \equiv \mathcal{C}\{\tilde{f}_\alpha, f_{{\rm M}\beta}\} = -\frac{\partial}{\partial \mathbf{v}} \boldsymbol{\cdot} \sum_\beta \mathbf{J}_v^{\alpha/\beta}, \tag{A1}$$

$$\mathbf{J}_v^{\alpha/\beta} = -\frac{m_\alpha}{m_\alpha + m_\beta}\,\nu_{\mathbf{p}}^{\alpha/\beta}\mathbf{v}\tilde{f}_\alpha - \frac{1}{4}\,\nu_\perp^{\alpha/\beta}(v^2\mathbf{I} - \mathbf{v}\mathbf{v})\boldsymbol{\cdot}\frac{\partial \tilde{f}_\alpha}{\partial \mathbf{v}} - \frac{1}{2}\,\nu_\parallel^{\alpha/\beta}\mathbf{v}\mathbf{v}\boldsymbol{\cdot}\frac{\partial \tilde{f}_\alpha}{\partial \mathbf{v}}. \tag{A2}$$

Here, $\mathbf{J}_v$ is the current (or flux) in velocity space. The Coulomb collision rates for momentum (subscript $\mathbf{p}$) loss $\nu_{\mathbf{p}}$ (which is due to dynamical friction, and often called "slowing down" and labeled with a subscript s), perpendicular (to $\mathbf{v}$ direction of α species) scattering $\nu_\perp$ and parallel (to $\mathbf{v}$) speed (v) diffusion $\nu_\parallel$, and the rate of speed reduction ν_v and energy loss ν_ε are

$$\nu_{\mathbf{p}}^{\alpha/\beta} = \left[\frac{m_\alpha + m_\beta}{m_\beta}\,\psi(x)\right]\nu_0, \tag{A3}$$

$$\nu_\perp^{\alpha/\beta} = 2\left[\left(1 - \frac{1}{2\,x}\right)\psi(x) + \psi'(x)\right]\nu_0, \tag{A4}$$

$$\nu_\parallel^{\alpha/\beta} = \left[\frac{\psi(x)}{x}\right]\nu_0, \tag{A5}$$

$$\nu_v^{\alpha/\beta} = \left[\frac{m_\alpha}{m_\beta}\psi - \frac{\psi}{2x} + \psi'\right]\nu_0 = \frac{\nu_\perp}{2} + \left(\frac{m_\alpha - m_\beta}{m_\alpha + m_\beta}\right)\nu_{\mathbf{p}}, \tag{A6}$$

$$\nu_\varepsilon^{\alpha/\beta} = 2\left[\frac{m_\alpha}{m_\beta}\psi - \psi'\right]\nu_0 = 2\,\nu_{\mathbf{p}} - \nu_\perp - \nu_\parallel. \tag{A7}$$

The reference Coulomb collision frequency $\nu_0 \to \nu_0^{\alpha/\beta}$ for collisions of species α charged particles with species β charged particles, the dimensionless kinetic energy $x \to x^{\alpha/\beta}$ and the Maxwell integral ψ and its derivative are

$$\nu_0^{\alpha/\beta} \equiv \frac{4\pi\, n_\beta\, q_\alpha^2 q_\beta^2 \ln\Lambda_{\alpha/\beta}}{\{4\pi\epsilon_0\}^2\, m_\alpha^2\, v^3}, \;\; x^{\alpha/\beta} \equiv \frac{m_\beta v^2}{2\,T_\beta} = \frac{v^2}{v_{T\beta}^2}, \tag{A8}$$

$$\psi(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x dt\, t^{1/2}e^{-t}, \;\; \psi' \equiv \frac{d\psi}{dx} = \frac{2\,x^{1/2}e^{-x}}{\sqrt{\pi}}. \tag{A9}$$

Adding all the species contributions in the collision rates using $\nu_{\mathbf{p}} \equiv \sum_\beta \nu_{\mathbf{p}}^{\alpha/\beta}$ etc. and taking account of their speed dependence, the F-P collision operator in (A1) can be written in various forms:

$$\begin{aligned}\mathcal{C}_{\rm t}\{\tilde{f}_\alpha\} &= \frac{\nu_\perp}{2}\,\pounds\{\tilde{f}_\alpha\} + \frac{1}{2v^2}\frac{\partial}{\partial v}\left[v^3\nu_\varepsilon\tilde{f}_\alpha + \frac{1}{v}\frac{\partial}{\partial v}(v^5\nu_\parallel\tilde{f}_\alpha)\right] \\ &= \frac{\nu_\perp}{2}\,\pounds\{\tilde{f}_\alpha\} + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^3\nu_0\psi\left(\frac{m_\alpha}{m_\beta}\tilde{f}_\alpha + \frac{\partial \tilde{f}_\alpha}{\partial x^{\alpha/\beta}}\right)\right] \\ &= \frac{\nu_\perp}{2}\,\pounds\{\tilde{f}_\alpha\} + \frac{\nu_\parallel}{2}v^2\frac{\partial^2\tilde{f}_\alpha}{\partial v^2} + \nu_v v\frac{\partial \tilde{f}_\alpha}{\partial v} + \nu_c\tilde{f}_\alpha.\end{aligned} \tag{A10}$$

Here, $\nu_c \equiv 2\,(m_\alpha/m_\beta)\,\nu_0\psi'/x^{\alpha/\beta}$ and $\pounds$ is the Lorentz scattering operator which in terms of the pitch angle ϑ and phase angle φ spherical velocity space coordinates is

$$\pounds\{\tilde{f}_\alpha\} \equiv \frac{1}{2}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial \tilde{f}_\alpha}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\,\frac{\partial^2\tilde{f}_\alpha}{\partial\varphi^2}\right]. \tag{A11}$$

The first line in (A10) indicates the effects of perpendicular and parallel (to $\mathbf{v}$) collisional scattering and energy loss that follow directly from the velocity-space current $\mathbf{J}_v$ in (A2). The second line shows that if the β species is the α species this collision operator relaxes $\tilde{f}_\alpha$ toward a Maxwellian where $\tilde{f}_\alpha \propto e^{-x^{\alpha/\alpha}} = e^{-v^2/v_{T\alpha}^2}$. Finally, the last line in (A10) provides the simplest formulation of $\perp$ and $\parallel$ diffusion, and speed reduction effects.

The relative importance of the various terms in the last line of (A10) can be estimated as follows. First, assume all the collision frequencies in (A3)–(A7) are comparable and use $\epsilon \equiv \nu/\omega \ll 1$ as a generic small parameter. Next, note that the time scale where Coulomb collisional effects on Landau damping are most important is $t \sim 1/\nu_{\rm eff}$. For a simple scaling assume that the $\nu_\perp$ term in (2) is dominant so $\nu_{\rm eff} \sim (\nu_\perp\omega)^{1/2} \sim \nu\,\epsilon^{-1/2} \gg \nu$. As shown in Fig. 1 and the attendant text, in a short time $t \sim 1/\nu_{\rm eff} \sim \epsilon^{1/2}/\nu \ll 1/\nu$ collisions only cause small diffusive spreads in the pitch-angle $\delta\vartheta \sim \sqrt{\nu_\perp t} \sim \epsilon^{1/4}$ and speed $\delta v/v \sim \sqrt{2\,\nu_\parallel t} \sim \epsilon^{1/4}$. Thus, estimating the derivatives in (A10) with $\partial/\partial\vartheta \sim 1/\delta\vartheta \sim \epsilon^{-1/4}$ and $v\,\partial/\partial v \sim v/\delta v \sim \epsilon^{-1/4}$, the scaling of the terms in the last line of (A10) is

$$\mathcal{C}_{\rm t}\{\tilde{f}_\alpha\} \sim \left(\frac{\nu_\perp}{\epsilon^{1/2}} + \frac{\nu_\parallel}{\epsilon^{1/2}} + \frac{\nu_v}{\epsilon^{1/4}} + \frac{\nu_c}{\epsilon^0}\right)\,\tilde{f}_\alpha. \tag{A12}$$

The parameter $\epsilon = \nu/\omega$ is very small; e.g., $\epsilon \sim 10^{-4}$ for the plasmas in Ref. [2] and it is often much smaller than this. Thus, in this ordering scheme the first two terms in (A12) are dominant by $\epsilon^{-1/4}$ ($\gtrsim 10$); i.e., the highest (second) order derivative terms in (A10) are dominant.

While the collision-induced speed reduction effect indicated by the ν_v terms in (A10) and (A12) is smaller than the higher derivative $\nu_\perp$ and $\nu_\parallel$ effects, it is of interest to consider its effect. If it was included in the analysis in the main text, its effects would be to change the first order speed characteristic equation from $dv/dt = 0$ to $dv/dt = -\nu_v v$ which yields $v = v_0 e^{-\nu_u\tau} \simeq v_0(1 - \nu_v\tau)$. In turn, this would change the first order deterministic particle trajectory determined from $d\mathbf{x}/dt = \mathbf{v}$ to $\mathbf{x} = \mathbf{x}_0 + \int_{t_0}^t dt'\,\mathbf{v}_0\,e^{-\nu_v t'} = \mathbf{x}_0 + \mathbf{v}_0\tau(1 - \nu_v\tau/2 + \cdots)$. For the ordering scheme used for (A12) these modifications of v_0 and $\mathbf{x}$ add terms of order $\epsilon^{1/4} \ll 1$ in the phase Φ, which are quite small and hence can be neglected.

The speed reduction effect can also be considered in the special limit where parallel diffusion dominates, i.e., $\epsilon_\parallel^{1/3} \gg \epsilon_\perp^{1/2}$, for which $\delta v/v_0 \sim \epsilon_\parallel^{1/3}$ and $v\,\partial/\partial v \sim \epsilon_\parallel^{-1/3}$. Then, using $v_0 \to v_0(1-\nu_v\tau)$ and $\delta v \to (v-v_0)+\nu_v\tau$, the phase Φ in (1) acquires two $\nu_v\tau$ terms that at $\tau \sim 1/\nu_{\rm eff} \sim \epsilon_\parallel^{1/3}/\omega$ are of order $\epsilon_\parallel^{1/3}\nu_v/\nu_\parallel$ and $\epsilon_\parallel^{1/3}$, which are again much less than unity and hence can be neglected. Their effect decreases the Δ_u where $\mathcal{I}m\{I_{\nu 0}\}$ peaks by a negligible amount $\sim \epsilon_\parallel^{2/3} \ll 1$, i.e., by $\epsilon_\parallel^{1/3}$ for $\Delta_u/\epsilon_\parallel^{1/3}$.

All these analyses have shown the $\nu_v v\,\partial\tilde{f}_\alpha/\partial v$ and $\nu_c\tilde{f}_\alpha$ terms in (A10) can be neglected. Thus, retaining only the highest order derivative terms in (A10) and neglecting the phase angle derivatives, which are unimportant for Landau damping because $\tilde{f}_\alpha$ does not depend on φ, for $t \lesssim 1/\nu_{\rm eff} \ll 1/\nu_0$ the approximate (subscript a) F-P operator obtained from (A10) plus (A11) becomes

$$\mathcal{C}_{\rm a}\{\tilde{f}_\alpha\} \simeq \frac{\nu_\perp}{4\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\,\frac{\partial\tilde{f}_\alpha}{\partial\vartheta}\right) + \frac{\nu_\parallel v^2}{2}\frac{\partial^2\tilde{f}_\alpha}{\partial v^2}. \quad \text{(A13)}$$

Specializing to a perturbed distribution composed of electrons as species α so $\tilde{f}_\alpha \to \tilde{f}_e$ plus ions of charge Z_i and to electrons on the tail of the lowest order electron Maxwellian distribution where $x^{e/e} = v^2/v_{T_e}^2 \gg 1$ and $x^{e/i} = v^2/v_{Ti}^2 \gg 1$ yields $\psi(x^{e/e}) \simeq 1$ and $\psi(x^{e/i}) \simeq 1$. Further, using the quasineutrality condition $n_iZ_i = n_e$ in the $\nu_0^{e/i}$ definition in (3) and $\nu_\perp = \nu_\perp^{e/e} + \nu_\perp^{e/i}$ yields

$$\nu_{\mathbf{p}} \simeq (2+Z_i)\,\nu_0^{e/e}, \quad \nu_\perp \simeq 2\,(1+Z_i)\,\nu_0^{e/e},$$
$$\nu_\parallel \simeq (v_{Te}/v)^2\nu_0^{e/e}, \quad \nu_v \simeq \nu_0^{e/e}, \quad \nu_\varepsilon \simeq 2\,\nu_0^{e/e}. \quad \text{(A14)}$$

Identifying $\nu \equiv \nu_0^{e/e}$, the various parameters in $\mathcal{C}_{\rm a}\{\tilde{f}_e\}$ in (A13) and hence (3) have now all been specified.

Appendix B: I_ν Integral For $\vartheta \neq 0$

The argument of the exponent in the $d\vartheta_0$ integral in the I_ν given in (15) can be rewritten as

$$-(\vartheta_0-\vartheta)^2/(2\,\epsilon_\perp z) + iz(\vartheta_0^2-\vartheta^2)/2$$
$$= -\frac{1-i\,\epsilon_\perp z^2}{2\,\epsilon_\perp z}\left[\vartheta_0 - \frac{\vartheta}{1-i\,\epsilon_\perp z^2}\right]^2 - \frac{\vartheta^2}{2}\,\frac{\epsilon_\perp z^3}{1-i\,\epsilon_\perp z^2}. \quad \text{(B1)}$$

Define $w \equiv [(1-i\,\epsilon_\perp z^2)/(2\,\epsilon_\perp z)]^{1/2}\,[\vartheta_0 - \vartheta/(1-i\,\epsilon_\perp z^2)]$ and $w_o \equiv \vartheta\,/\,[(2\,\epsilon_\perp z)\,(1-i\,\epsilon_\perp z^2)]$. This yields $\vartheta_0 = w\,[(2\,\epsilon_\perp z)/(1-i\,\epsilon_\perp z^2)]^{1/2} + w_o$, and the ϑ_0 integral is

$$I_\vartheta(\vartheta,\epsilon_\perp,z) \equiv \int_0^\infty \frac{\vartheta_0 d\vartheta_0}{\epsilon_\perp z} e^{-(\vartheta_0-\vartheta)^2/(2\,\epsilon_\perp z)+iz(\vartheta_0^2-\vartheta^2)/2}$$
$$= \frac{e^{-(\vartheta^2/2)\,\epsilon_\perp z^3/(1-i\,\epsilon_\perp z^2)}}{1-i\epsilon_\perp z^2}\Big(e^{-w_o^2} + \sqrt{\pi}w_o[1+\mathrm{erf}(w_o)]\Big). \quad \text{(B2)}$$

For $\vartheta = 0$, which causes $w_o = 0$, this yields the simple result $I_\vartheta(\vartheta\!=\!0) = 1/(1-i\,\epsilon_\perp z^2)$.

The argument of the exponent in the $\delta v \equiv v_0 - v$ integral in the I_ν defined in (15) can be rewritten as

$$-(\delta v/v_0)^2/(4\epsilon_\parallel z) - iz\,\delta v/v_0$$
$$= -\left[\frac{\delta v/v_0}{\sqrt{4\epsilon_\parallel z}} + i\,\epsilon_\parallel^{1/2}z^{3/2}\right]^2 - \epsilon_\parallel z^3. \quad \text{(B3)}$$

Defining $y \equiv (\delta v/v_0)/\sqrt{4\epsilon_\parallel z} + i\,\epsilon_\parallel^{1/2}z^{3/2}$, the δv integral becomes $(e^{-\epsilon_\parallel z^3}/\sqrt{\pi})\int_{-\infty}^{\infty} dy\, e^{-y^2} = e^{-\epsilon_\parallel z^3}$.

Using the results of these two integrals in (15) yields

$$I_\nu \equiv \int_0^{kut} dz\; i\,e^{-i\Delta_u z-\epsilon_\parallel z^3}\; I_\vartheta(\vartheta,\epsilon_\perp,z) \quad \text{(B4)}$$

$$\overset{\vartheta=0}{\Longrightarrow} \quad I_{\nu 0} \equiv \int_0^{kut} dz\; \frac{i\,e^{-i\Delta_u z-\epsilon_\parallel z^3}}{1-i\,\epsilon_\perp z^2}. \quad \text{(B5)}$$

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