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Transport equations in tokamak plasmas

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Tokamak plasma transport equations are usually obtained by flux surface averaging the collisional Braginskii equations. However, tokamak plasmas are not in collisional regimes. Also, *ad hoc* terms are added for: neoclassical effects on the parallel Ohm's law; fluctuation-induced transport; heating, current-drive and flow sources and sinks; small magnetic field non-axisymmetries; magnetic field transients etc. A set of self-consistent second order in gyroradius fluid-moment-based transport equations for nearly axisymmetric tokamak plasmas has been developed using a kinetic-based approach. The derivation uses neoclassical-based parallel viscous force closures, and includes all the effects noted above. Plasma processes on successive time scales and constraints they impose are considered sequentially: compressional Alfvén waves (Grad-Shafranov equilibrium, ion radial force balance); sound waves (pressure constant along field lines, incompressible flows within a flux surface); and collisions (electrons, parallel Ohm's law; ions, damping of poloidal flow). Radial particle fluxes are driven by the many second order in gyroradius toroidal angular torques on a plasma species: 7 ambipolar collision-based ones (classical, neoclassical etc.) and 8 non-ambipolar ones (fluctuation-induced, polarization flows from toroidal rotation transients etc.). The plasma toroidal rotation equation results from setting to zero the net radial current induced by the non-ambipolar fluxes. The radial particle flux consists of the collision-based intrinsically ambipolar fluxes plus the non-ambipolar fluxes evaluated at the ambipolarity-enforcing toroidal plasma rotation (radial electric field). The energy transport equations do not involve an ambipolar constraint and hence are more directly obtained. The "mean field" effects of microturbulence on the parallel Ohm's law, poloidal ion flow, particle fluxes, and toroidal momentum and energy transport are all included self-consistently. The final equations describe radial transport of plasma toroidal rotation, and poloidal and toroidal magnetic fluxes, as well as the usual particle and energy transport.

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I. INTRODUCTION

Transport equations for describing tokamak plasmas were initially advanced in the late 1970s. They were developed by adapting the collisional Braginskii [1] transport equations for plasma density and energy transport to the tokamak toroidal geometry. In addition, many other terms have been added over the years in an *ad hoc* manner to include other processes important in tokamak plasmas: neoclassical effects on the parallel Ohm's law [2, 3] (trapped particle effects on the resistivity, bootstrap current), "anomalous" plasma transport induced by microturbulence, auxiliary heating, current-drive and flow sources and sinks, effects of small magnetic field departures from axisymmetry etc. The resultant transport equations form the basis for ONETWO [4], TRANSP [5] and other transport modeling codes that are used to model present and future tokamak plasma experiments.

However, most tokamak plasmas are not in a collisional regime — collision lengths are typically much longer than the toroidal circumference of the tokamak. More rigorous and self-consistent plasma transport equations need to be developed from a kinetic-based approach. And all the effects indicated in the preceding paragraph need to be included naturally and self-consistently.

In this paper we develop [6, 7] self-consistent radial plasma transport equations from velocity-space (fluid) moments of a kinetic-based approach that includes all these effects for nearly axisymmetric tokamak plasmas using neoclassical-based parallel viscous force closures [3]. This new approach accounts for lower order radial force balance constraints and solves for the flows within flux surfaces before determining the net radial transport equations. This procedure is analogous to that used for determining radial neoclassical plasma transport equations in non-axisymmetric stellarator plasmas — see for example Ref. [8] and references cited therein. This new approach was originally developed [6, 7] for determining the toroidal rotation in tokamak plasmas. This paper extends that analysis to include heat transport equations and takes into account the poloidal heat flows that were neglected there. For additional details see Refs. [6, 7].

This paper is organized as follows. The next section discusses the fundamental kinetic equation from which the analysis begins and the resultant fluid moment equations that are used. Also, the key assumptions, small gyroradius expansion used for ordering various terms and lowest order radial force balance considerations are presented there. The following section (III) describes the determination of the first order plasma flows within flux surfaces. The transient evolution of poloidal and toroidal magnetic fluxes and their effects on plasma transport are described in Section IV. The radial particle fluxes induced by the many toroidal torques on the plasma [7]

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are summarized in Section V. Also, the toroidal plasma rotation equation for a tokamak [6, 7] obtained by setting the net radial current to zero to enforce plasma quasineutrality is summarized there. This key equation also determines the radial electric field and the resultant ambipolar radial particle flux. Section VI develops the electron and ion energy transport equations. Finally, the results of this paper are summarized in Section VII which also discusses some key new features of this comprehensive transport model.

II. BASIC EQUATIONS AND ASSUMPTIONS

We begin with a plasma kinetic equation (PKE) that includes the Fokker-Planck Coulomb collision operator $\mathcal{C}\{f\}$ and kinetic sources $S\{f\}$ for each plasma species:

$$\left.\frac{\partial f}{\partial t}\right|_{\mathbf{x}} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial \mathbf{v}} = \mathcal{C}\{f\} + S\{f\}. \quad (1)$$

Fluid moment equations for the species density n, flow velocity $\mathbf{V}$, temperature T and conductive heat flow $\mathbf{q}$ are obtained by taking velocity-space moments $\int d^3v\,[1, m\mathbf{v}, mv^2/2, \mathbf{v}(mv^2/2)]$ of this PKE:

$$\left.\frac{\partial n}{\partial t}\right|_{\mathbf{x}} + \nabla\cdot n\mathbf{V} = S_n, \quad (2)$$

$$\left.\frac{\partial}{\partial t}\right|_{\mathbf{x}}(mn\mathbf{V}) + \nabla\cdot(mn\mathbf{V}\mathbf{V}) = nq\,(\mathbf{E}+\mathbf{V}\times\mathbf{B}) - \nabla p - \nabla\cdot\boldsymbol{\pi} + \mathbf{R}_{\rm V} + \mathbf{S}_{\rm V}, \quad (3)$$

$$\frac{3}{2}\left.\frac{\partial p}{\partial t}\right|_{\mathbf{x}} + \nabla\cdot\left(\mathbf{q}+\frac{5}{2}p\mathbf{V}\right) = Q + \mathbf{V}\cdot\nabla p - \boldsymbol{\pi}:\nabla\mathbf{V} + S_E^{\ddagger}, \quad (4)$$

$$\left.\frac{\partial \mathbf{q}}{\partial t}\right|_{\mathbf{x}} = \frac{q}{m}\mathbf{q}\times\mathbf{B} - \frac{T}{m}\left(\frac{5}{2}n\nabla T + \nabla\cdot\boldsymbol{\Theta} - \mathbf{R}_{\rm q}\right) - \nabla\cdot\left(\mathbf{V}\mathbf{q}+\mathbf{q}\mathbf{V}+\frac{2}{3}\mathbf{V}\cdot\mathbf{q}\,\mathbf{I}\right) + \frac{5}{2}p\mathbf{V}\cdot\nabla\mathbf{V} + \mathbf{S}_{\rm q}^{\ddagger} + \cdots. \quad (5)$$

As indicated, in this conservative form of these equations the fluid moment properties are evaluated at a laboratory position $\mathbf{x}$. Here, the species pressure is $p \equiv nT$, and $\mathbf{R}_{\rm V}$ and $\mathbf{R}_{\rm q}$ are the collisional friction and heat friction forces. The density, momentum, net energy and net heat flow sources are S_n, $\mathbf{S}_{\rm V}$, $S_E^{\ddagger} \equiv S_E - \mathbf{V}\cdot\mathbf{S}_V - (mV^2/2)S_n$ and $\mathbf{S}_{\rm q}^{\ddagger} \equiv \mathbf{S}_{\rm q} - (5T/2m)(\mathbf{S}_{\rm V} - m\mathbf{V}S_n)$. The $\cdots$ at the end of (5) indicates higher order terms that will be neglected here. The remaining notation and definitions are relatively standard [1–3]. The fluid moments required to close these equations are the stress tensor $\boldsymbol{\pi}$ and heat stress tensor $\boldsymbol{\Theta}$; the needed parallel viscous forces they induce are given in Section III.

A number of assumptions are made to facilitate the analysis: 1) Particle gyroradii are small which to zeroth order yields magnetohydrodynamic (MHD) force balance equilibrium and the radial ion force balance, flows on flux surfaces at first order and second order "radial" transport fluxes. 2) Magnetic flux surfaces are nested and toroidally axisymmetric to lowest order (i.e., there are no magnetic islands in the region of interest). 3) Non-axisymmetries (NA) in the magnetic field are first order in the gyroradius, which causes the toroidal flow damping rate to be one order smaller than the poloidal flow damping rate. 4) Collision lengths are long compared to the plasma toroidal circumference so plasma properties are constant on magnetic flux surfaces, which is valid for most tokamaks except possibly in a cold plasma edge. 5) Electron and ion distribution functions are Maxwellian to lowest order in the gyroradius expansion (e.g., there are no radio-frquency-wave-induced zeroth order distribution distortions); 6) Microinstability-induced plasma fluctuations are gyroradius small and lead mostly to second order "anomalous" plasma transport across flux surfaces. 7) Impurity flow velocities are approximately equal to hydrogenic ion velocities, which simplifies inclusion of impurity effects. Finally, 8) magnetic field transients are slow enough that they only occur on the plasma transport or longer time scale.

A small gyroradius expansion is used to order the various terms in the fluid moment equations: $\delta \equiv \varrho/a \ll 1$ in which ϱ is the particle gyroradius in the magnetic field and a is the minor radius of the plasma. For example, the plasma pressure is expanded as

$$p(\mathbf{x}) = p_0(\rho) + \delta\,[\bar{p}_1(\rho,\theta) + \tilde{p}_1(\rho,\theta,\zeta)] + \mathcal{O}\{\delta^2\}. \quad (6)$$

The lowest order species pressure p_0 is only a function of the flux surface label ρ. The first order toroidal-angle-averaged pressure $\bar{p}_1(\rho,\theta)$ represents the Pfirsch-Schlüter poloidal (θ) pressure variation on a flux surface. Finally, $\tilde{p}_1(\rho,\theta,\zeta)$ represents the gyroradius-small non-axisymmetric perturbations induced by externally imposed magnetic fields or collective plasma instabilities.

Because of lowest order axisymmetry in the toroidal angle ζ, perturbations will be expanded in a Fourier series: $\tilde{p}_1 = \sum_n \hat{p}_n e^{-in\zeta}$, $\hat{p}_n \equiv (1/2\pi)\int_0^{2\pi} d\zeta\, e^{in\zeta}\,\tilde{p}_1$. The $n=0$ term defines the average over toroidal angle:

$$\overline{p(\mathbf{x})} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\zeta\, p(\mathbf{x}) = p_0(\rho) + \delta\,\bar{p}_1(\rho,\theta) + \mathcal{O}\{\delta^2\}. \quad (7)$$

Perpendicular scale lengths of fluctuations will be presumed to be on the gyroradius scale. Hence, perpendicular derivatives of fluctuations will be assumed to be large: $\nabla_\perp \tilde{p}_1 \sim (1/\delta)\,\delta \sim \delta^0$. But parallel derivatives of fluctuations will be assumed to be on the macroscopic ($\sim a$) scale and hence small: $\nabla_\parallel \tilde{p}_1 \sim \delta^0\delta \sim \delta$. Derivatives of equilibrium components p_0, $\bar{p}_1$ will be $\mathcal{O}\{\delta^0,\delta\}$.

The magnetic field will be represented by an average $\mathbf{B}_0 \equiv \mathbf{B}_t + \mathbf{B}_p = I\nabla\zeta + \nabla\zeta\times\nabla\psi_p = \nabla\psi_p\times\nabla(q\theta-\zeta) = \nabla\times(\psi_t\nabla\theta - \psi_p\nabla\zeta)$ plus small $\mathcal{O}\{\delta\}$ perturbations $\tilde{\mathbf{B}}$ due to 3D NA and collective responses: $\mathbf{B} = \mathbf{B}_0(\rho,\theta) + \delta\,(\tilde{\mathbf{B}}_\parallel + \tilde{\mathbf{B}}_\perp) + \mathcal{O}\{\delta^2\}$, $|\mathbf{B}| \simeq B_0(\rho,\theta) + \delta\,\tilde{B}_\parallel + \mathcal{O}\{\delta^2\}$. Here, $I(\psi_p) \equiv RB_t$ and ψ_p, ψ_t are the poloidal, toroidal magnetic fluxes. The electric field will be represented as a sum of scalar and vector potentials: $\mathbf{E} = -\nabla\phi + \mathbf{E}^A$, in which the inductive electric field is $\mathbf{E}^A \equiv -\partial\mathbf{A}/\partial t$. Since the ζ-average vector potential is $\overline{\mathbf{A}} = \psi_t \boldsymbol{\nabla}\theta - \psi_p \boldsymbol{\nabla}\zeta$, the ζ-average toroidal inductive electric field will be written as $\overline{\mathbf{E}}^A = (\partial\Psi/\partial t + \dot{\psi}_p)\boldsymbol{\nabla}\zeta - \dot{\psi}_t \boldsymbol{\nabla}\theta \sim \mathcal{O}\{\delta^2\}$, in which $2\pi\partial\Psi/\partial t = V^{\zeta}_{\rm loop}(t)$ is the toroidal loop voltage and the dots over the magnetic fluxes indicate their partial time derivatives at a given laboratory position $\mathbf{x}$.

The lowest order (δ^0) fluid moment equations in (2)–(4) yield a two-fluid form of the ideal MHD model. Compressional Alfvén waves perpendicular to the magnetic field enforce $\mathbf{J}_0\times\mathbf{B}_0 = \boldsymbol{\nabla} P_0$ on the fast Alfvén time scale. This relation in combination with a two-fluid Ohm's law $\mathbf{E}_0 + \mathbf{V}\times\mathbf{B}_0 = (\mathbf{J}_0\times\mathbf{B}_0 - \boldsymbol{\nabla} p_e)/n_e e$ yields the equilibrium ($t > 1/\nu_e$, neglecting electron inertia) radial force balance $0 = \mathbf{e}_\rho \cdot [n_i q_i(\mathbf{E} + \mathbf{V}\times\mathbf{B}) - \boldsymbol{\nabla} p_i]$, which in our toroidal coordinates yields [6, 7]

$$\Omega_t \equiv \mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}\zeta = -\left(\frac{d\Phi}{d\psi_p} + \frac{1}{n_i q_i}\frac{dp_i}{d\psi_p} - q\,\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}\theta\right). \quad (8)$$

In cyclindrical-type coordinates ($d\psi_p \simeq B_p R\, d\rho$) this relation between toroidal and poloidal flows, the radial electric field and radial ion pressure gradient for each ion species is $V_t \simeq E_\rho/B_p - [1/(n_i q_i)]\,(dp_i/d\rho) + (B_t/B_p)V_p$.

Maxwellianization of the electron, ion distributions on their collision times of $1/\nu_e, 1/\nu_i$ cause n, T to be constant over collision lengths λ_e, λ_i and hence on flux surfaces. Also, species flow velocities $\mathbf{V}_s$ become physically meaningful on the collision time scales. As discussed in the following section, to first order (δ) in the small gyroradius expansion the plasma flows are on flux surfaces. The radial flows across flux surfaces are second order (δ^2).

III. CURRENTS, FLOWS ON FLUX SURFACES

This work extends our previous analyses of flows [6, 7] to include heat flow effects. At order δ the flows are on magnetic flux surfaces and can be represented by components in the θ, ζ or $\|, \wedge$ directions [3, 7]:

$$\overline{\mathbf{V}}_1 \equiv \mathbf{e}_\theta(\overline{\mathbf{V}}\boldsymbol{\cdot}\boldsymbol{\nabla}\theta) + \mathbf{e}_\zeta(\overline{\mathbf{V}}\boldsymbol{\cdot}\boldsymbol{\nabla}\zeta) = \overline{V}_\|\mathbf{B}_0/B_0 + \overline{\mathbf{V}}_\wedge. \quad (9)$$

The conductive heat flow is represented similarly. The cross flow in (9) indicates $\mathbf{E}_0\times\mathbf{B}_0$ and diamagnetic flows:

$$\overline{\mathbf{V}}_{s\wedge} \equiv \frac{\mathbf{B}_0\times\boldsymbol{\nabla}\psi_p}{B_0^2}\left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_p}\right). \quad (10)$$

To lowest order (5) yields the diamagnetic cross heat flow $\overline{\mathbf{q}}_{s\wedge} = (5 n_{s0} T_{s0}/2 q_s)(\mathbf{B}_0\times\boldsymbol{\nabla}\psi_p/B_0^2)(dT_{s0}/d\psi_p)$.

On transport time scales the first order flows and heat flows are incompressible due to sound wave equilibration along field lines. Then, the poloidal flow function U_θ becomes constant on a flux surface and is given by [3, 7]

$$U_\theta(\psi_p) \equiv \frac{\overline{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\theta}{\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta} = \frac{\mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{V}}_1}{B_0^2} + \frac{\overline{\mathbf{V}}_\wedge\boldsymbol{\cdot}\boldsymbol{\nabla}\theta}{\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta}. \quad (11)$$

The poloidal heat flow function is similarly represented: $Q_{s\theta}(\psi_p) \equiv (-2/5 n_{s0} T_{s0})\,\overline{\mathbf{q}}_s\boldsymbol{\cdot}\boldsymbol{\nabla}\theta/\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta$. The poloidal component of the cross flow for species s is

$$\frac{\overline{\mathbf{V}}_{s\wedge}\boldsymbol{\cdot}\boldsymbol{\nabla}\theta}{\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta} = \frac{I}{B_0^2}\left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_p}\right). \quad (12)$$

The poloidal cross heat flow is $\overline{\mathbf{q}}_{s\wedge}\boldsymbol{\cdot}\boldsymbol{\nabla}\theta/\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta = (5/2)(I n_{s0} T_{s0}/q_s B_0^2)(dT_{s0}/d\psi_p)$. Thus, we obtain

$$Q_{s\theta}(\psi_p) = \frac{-2}{5 n_{s0} T_{s0} B_0^2}\mathbf{B}_0\boldsymbol{\cdot}\,\overline{\mathbf{q}}_s - \frac{I}{q_s B_0^2}\frac{dT_{s0}}{d\psi_p}. \quad (13)$$

Multiplying (11) by $n_{s0} q_s B_0^2$, summing over species s and flux surface averaging (FSA) yields [7] $\langle B_0^2\rangle K_J \equiv \sum_s n_{s0} q_s \langle B_0^2\rangle U_{s\theta} = \langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}\rangle + I\, dP_0/d\psi_p$ in which $P_0(\psi_p) \equiv \sum_s p_{s0}$ is the total plasma pressure. Hence for the total plasma parallel current we obtain

$$B_0 J_\| = \frac{\langle\mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}\rangle B_0^2}{\langle B_0^2\rangle} - I\frac{dP_0}{d\psi_p}\left(1 - \frac{B_0^2}{\langle B_0^2\rangle}\right). \quad (14)$$

The first term represents the FSA parallel current contribution while the second indicates the Pfirsch-Schlüter current. In tokamaks the FSA parallel current is defined in terms of the poloidal flux ψ_p by [2, 3, 7]

$$\mu_0\langle\mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}\rangle = I\,\langle R^{-2}\rangle\,\Delta^+\psi_p, \quad (15)$$

in which the second order cylindrical-type operator is

$$\Delta^+\psi_p \equiv \frac{I}{\langle R^{-2}\rangle V'}\frac{\partial}{\partial\rho}\left[\left\langle\frac{|\boldsymbol{\nabla}\rho|^2}{R^2}\right\rangle\frac{V'}{I}\frac{\partial\psi_p}{\partial\rho}\right]. \quad (16)$$

Here and hereafter $V' \equiv dV/d\rho$ is the derivative of the volume $V(\rho) \equiv \int_0^\rho d^3x$ of the ρ flux surface.

Flows on flux surfaces in tokamak plasmas are obtained by solving coupled momentum and heat flow equations for electrons and hydrogenic and impurity ions [3]. For example, they can be evaluated using the NCLASS code [9]. However, assuming, as is often the case, impurities are in the plateau or Pfirsch-Schlüter collisionality regimes, the flow velocities of impurities are approximately equal to those of the hydrogenic ions and an analytic analysis is possible [10]. Making this asssumption facilitates the straightforward inclusion of impurities, non-inductive sources, dynamo and other current and flow drives in the parallel Ohm's law and poloidal ion flow equations.

The parallel Ohm's law is obtained from the equilibrium ($t > 1/\nu_e$) first order FSA parallel electron momentum (3) and heat flow (5) equations [3, 10, 11]:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\,\overline{\boldsymbol{\pi}}_{e\|}\rangle + \langle\mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{R}}_{e\mathrm{V}}\rangle + \langle\mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{F}}_{e\mathrm{V}}\rangle \\ -\langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\,\overline{\boldsymbol{\Theta}}_{e\|}\rangle + \langle\mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{R}}_{eq}\rangle + \langle\mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{F}}_{eq}\rangle \end{bmatrix}. \quad (17)$$

The FSA parallel viscous stress and heat stress forces are

$$\begin{bmatrix} \langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\boldsymbol{\pi}_{e\|}\rangle \\ \langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\,\Theta_{e\|}\rangle \end{bmatrix} \equiv \frac{m_e n_e}{\tau_{ee}}\,\mathsf{M}_e\boldsymbol{\cdot}\begin{bmatrix} \langle B_0^2\rangle\,U_{e\theta} \\ \langle B_0^2\rangle\,Q_{e\theta} \end{bmatrix}. \quad (18)$$

The collisional friction and heat friction forces are [3]

$$\begin{bmatrix} \langle \mathbf{B}_0\cdot\mathbf{R}_{e\mathrm{V}}\rangle \\ \langle \mathbf{B}_0\cdot\mathbf{R}_{e\mathrm{q}}\rangle \end{bmatrix} = -\,\frac{m_e n_e}{\tau_{ee}}\,\mathsf{N}_e\cdot\begin{bmatrix} \frac{-1}{n_e e}\,\langle \mathbf{B}_0\cdot\mathbf{J}\rangle \\ \frac{-2}{5n_e T_e}\,\langle \mathbf{B}_0\cdot\mathbf{q}_e\rangle \end{bmatrix}. \quad (19)$$

The reference electron-electron collision frequency is [3]

$$\frac{1}{\tau_{ee}} \equiv \frac{4}{3\sqrt{\pi}}\,\frac{4\pi\, n_e e^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2\, m_e^2\, v_{Te}^3} \simeq \frac{5\times10^{-11}\, n_e(\mathrm{m}^{-3})}{[T_e(\mathrm{eV})]^{3/2}}. \quad (20)$$

In the asymptotic ($\nu_{*e} \ll 1$) banana collisionality regime the symmetric matrix of dimensionless electron viscosity coefficients is [3, 10, 11]

$$\mathsf{M}_e \equiv \begin{bmatrix} \mu_{e00} & \mu_{e01} \\ \mu_{e01} & \mu_{e11} \end{bmatrix} = \frac{f_t}{f_c}\begin{bmatrix} 0.53 + Z_{\mathrm{eff}} & 0.62 + \frac{3}{2}Z_{\mathrm{eff}} \\ 0.62 + \frac{3}{2}Z_{\mathrm{eff}} & 1.39 + \frac{13}{4}Z_{\mathrm{eff}} \end{bmatrix}. \quad (21)$$

Here, $f_t \equiv 1 - f_c$ is the fraction of trapped particles [3] in which $f_c \equiv (3/4)\langle B_0^2\rangle \int_0^{1/B_{\max}} \lambda\, d\lambda / \langle \sqrt{1-\lambda B_0(\theta)}\,\rangle \simeq 1 - 1.46\sqrt{\epsilon} + 0.46\,\epsilon\sqrt{\epsilon}$ [10] is the fraction of circulating particles. For an impure plasma $Z_{\mathrm{eff}} = \sum_i n_i Z_i^2/n_e$ is the effective ion charge. Multi-collisionality forms of M_e are given in [3, 10, 11]. The symmetric matrix of electron friction force coefficients is [3, 10, 11]

$$\mathsf{N}_e \equiv \begin{bmatrix} \nu_{e00} & \nu_{e01} \\ \nu_{e01} & \nu_{e11} \end{bmatrix} = \begin{bmatrix} Z_{\mathrm{eff}} & \frac{3}{2}Z_{\mathrm{eff}} \\ \frac{3}{2}Z_{\mathrm{eff}} & \sqrt{2} + \frac{13}{4}Z_{\mathrm{eff}} \end{bmatrix}. \quad (22)$$

Finally, the FSA parallel "external" forces and heat forces on the electrons are

$$\begin{bmatrix} \langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{V}}\rangle \\ \langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{q}}\rangle \end{bmatrix} = \begin{bmatrix} -n_{e0}e\langle \mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle + \langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{V}}^{\mathrm{tot}}\rangle \\ (m_e/T_{e0})\langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{q}}^{\mathrm{tot}}\rangle \end{bmatrix}. \quad (23)$$

The sources of electron momentum and heat flow are composed of both direct and fluctuation-induced sources: $\langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{V}}^{\mathrm{tot}}\rangle \equiv \langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{V}}^{\ddagger}\rangle + \langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{V}}^{\mathrm{fl}}\rangle$ and $\langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{q}}^{\mathrm{tot}}\rangle \equiv \langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{q}}^{\ddagger}\rangle + \langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{q}}^{\mathrm{fl}}\rangle$. The net electron non-inductive momentum source is $\langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{V}}^{\ddagger}\rangle = \langle \mathbf{B}_0\cdot(\overline{\mathbf{S}}_{e\mathrm{V}} - m_e\overline{\mathbf{V}}_e\overline{S}_{en})\rangle$ and $\langle \mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\mathrm{q}}^{\ddagger}\rangle$ is the net non-inductive source of electron heat flow. The fluctuation-induced electron momentum source is $\langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{V}}^{\mathrm{fl}}\rangle \equiv -\langle \mathbf{B}_0\cdot(m_e n_{e0}\overline{\tilde{\mathbf{V}}_e\cdot\boldsymbol{\nabla}\tilde{\mathbf{V}}_e} + \overline{\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{e\wedge}})\rangle - n_{e0}e\langle \mathbf{B}_0\cdot\overline{\tilde{\mathbf{V}}_{e\wedge}\times\tilde{\mathbf{B}}_\perp}\rangle$ and the analogous heat flow source is $\langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{q}}^{\mathrm{fl}}\rangle \equiv -\langle \mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\Big(\overline{\tilde{\mathbf{V}}_e\tilde{\mathbf{q}}_e + \tilde{\mathbf{q}}_e\tilde{\mathbf{V}}_e + (2/3)\tilde{\mathbf{V}}_e\cdot\tilde{\mathbf{q}}_e\mathbf{I}}\Big)\rangle + (5p_{e0}/2)\langle \mathbf{B}_0\cdot\overline{\tilde{\mathbf{V}}_e\cdot\boldsymbol{\nabla}\tilde{\mathbf{V}}_e}\rangle$, which involves many apparently small, rarely calculated or measured quantities.

The fluctuating electron flow $\tilde{\mathbf{V}}_e$ and heat flow $\tilde{\mathbf{q}}_e$ here are fluctuating fluid flows in the laboratory frame of reference. As discussed in Appendix A of Ref. [7], the species fluid flow $\mathbf{V}$ is related to the guiding center flow $\mathbf{V}_g \equiv \int d^3v\,\bar{\mathbf{v}}_d f/n$ induced by the guiding center drift velocity $\bar{\mathbf{v}}_d$ that is usually calculated in drift-kinetic and gyrokinetic calculations through [1] $\mathbf{V} = \mathbf{V}_g + (1/nq)\boldsymbol{\nabla}\times\mathbf{M}$. Here, $(1/nq)\boldsymbol{\nabla}\times\mathbf{M} \simeq \mathbf{V}_* = \mathbf{B}_0\times\boldsymbol{\nabla}p/nqB^2$ is the diamagnetic flow induced by the plasma magnetization $\mathbf{M} \equiv -p\mathbf{B}/B^2$ produced by the magnetic moments of the charged particles in the plasma. The fluid heat flow $\mathbf{q}$ is related to the guiding center heat flow similarly.

The electron parallel viscous force can be written in terms of the parallel current using the electron poloidal flow function in the form

$$n_e e\langle B_0^2\rangle U_{e\theta} = -\,\langle \mathbf{B}_0\cdot\mathbf{J}\rangle + n_e e\langle B_0^2\rangle U_{i\theta} - I\,\frac{dP_0}{d\psi_p}. \quad (24)$$

Using this result and the definitions in the preceding paragraphs, the matrix equation in (17) becomes

$$\frac{m_e}{e\tau_{ee}}\,[\,\mathsf{M}_e + \mathsf{N}_e]\cdot\begin{bmatrix} \langle \mathbf{B}_0\cdot\mathbf{J}\rangle \\ \frac{2e}{5T_e}\langle \mathbf{B}_0\cdot\mathbf{q}_e\rangle \end{bmatrix} = -\begin{bmatrix} \langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{V}}\rangle \\ \langle \mathbf{B}_0\cdot\overline{\mathbf{F}}_{e\mathrm{q}}\rangle \end{bmatrix} + \frac{1}{n_e e}\,\mathsf{M}_e\cdot\begin{bmatrix} -I\,dP_0/d\psi_p + n_e e\langle B_0^2\rangle U_{i\theta} \\ I\,n_e dT_{e0}/d\psi_p \end{bmatrix}. \quad (25)$$

This matrix equation can be solved for the parallel current and electron heat flow induced by the parallel "external" and viscous forces by inverting the $[\mathsf{M}_e + \mathsf{N}_e]$ matrix.

The parallel current solution of (25) can be written in the form of an extended neoclassical parallel Ohm's law:

$$\langle \mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle = \eta_\parallel^{\mathrm{nc}}\,(\,\langle \mathbf{B}_0\cdot\mathbf{J}\rangle - \langle \mathbf{B}_0\cdot\mathbf{J}_{\mathrm{drives}}\rangle\,)\,. \quad (26)$$

Here, the neoclassical parallel resistivity is

$$\eta_\parallel^{\mathrm{nc}} \equiv \frac{m_e}{n_e e^2\tau_{ee}}\,\frac{1}{[\mathsf{N}_e + \mathsf{M}_e]_{00}^{-1}} \geq \frac{m_e}{n_e e^2\tau_{ee}}\,\frac{1}{[\mathsf{N}_e]_{00}^{-1}} \equiv \frac{1}{\sigma_\parallel^{\mathrm{Sp}}}. \quad (27)$$

As indicated, when viscosity effects are negligible, this is just the reciprocal of the Spitzer electrical conductivity: $1/\sigma_\parallel^{\mathrm{Sp}} = (m_e\nu_e/n_{e0}e^2)\,(\sqrt{2} + Z_{\mathrm{eff}})/[\sqrt{2} + (13/4)Z_{\mathrm{eff}}]$ in which $\nu_e \equiv Z_{\mathrm{eff}}/\tau_{ee}$. This formula for the Spitzer conductivity is typically accurate to within about 1 % for $Z_{\mathrm{eff}} \sim$ 1–4, but incorrect by about 5 % for $Z_{\mathrm{eff}} \to \infty$. Greater accuracy can be obtained by including "energy-weighted heat flow" effects and inverting the resultant 3×3 matrix equation. However, such effort is not warranted because the intrinsic uncertainty in the Coulomb collision operator is $\sim 1/\ln\Lambda \sim 1/17 \simeq 6$ %, which is larger than errors in (27). Neglecting electron heat flow effects, the N_e and M_e matrices reduce to their 00 elements. Then, the neoclassical parallel resistivity becomes simply $\eta_\parallel^{\mathrm{nc}} \simeq \eta_\perp(1 + \mu_{e00}/\nu_{e00})$, in which $\eta_\perp \equiv m_e\nu_e/n_{e0}e^2$ is the perpendicular resistivity and for $\sqrt{\epsilon} \ll 1$ in the banana collisionality regime $\mu_{e00}/\nu_{e00} \sim 1.46\sqrt{\epsilon}\,(0.53 + Z_{\mathrm{eff}})/Z_{\mathrm{eff}}$.

Currents driven by the total radial pressure gradient (bootstrap current) and electron momentum sources are:

$$\langle \mathbf{B}_0\cdot\mathbf{J}_{\mathrm{drives}}\rangle \equiv \langle \mathbf{B}_0\cdot(\mathbf{J}_{\mathrm{bs}} + \mathbf{J}_{\mathrm{CD}} + \mathbf{J}_{\mathrm{dyn}})\,\rangle. \quad (28)$$

The bootstrap (bs), non-inductive current-drive (CD) and dynamo (dyn) currents are defined to be [7]

$$\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm bs}\rangle \;=\; b_{00}\left(-\,I\frac{dP_0}{d\psi_p}+n_e e\langle B_0^2\rangle U_{i\theta}\right) + b_{01}\left(n_{e0} I\frac{dT_e}{d\psi_p}\right), \qquad (29)$$

$$\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm CD}\rangle \;=\; -\,\frac{\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\ddagger}_{e{\rm V}}\rangle + c_{01}(m_e/T_{e0})\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\ddagger}_{e{\rm q}}\rangle}{n_{e0}e\,\eta_\parallel^{\rm nc}}, \qquad (30)$$

$$\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm dyn}\rangle \;=\; -\,\frac{\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{F}}{}^{\rm fl}_{e{\rm V}}\rangle + c_{01}(m_e/T_{e0})\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{F}}{}^{\rm fl}_{e{\rm q}}\rangle}{n_{e0}e\eta_\parallel^{\rm nc}}. \qquad (31)$$

Here, the coefficients are $b_{00} \equiv \big[[\mathsf{N}_e+\mathsf{M}_e]^{-1}\boldsymbol{\cdot}\,\mathsf{M}_e\big]_{00}$, $b_{01}\equiv\big[\,[\mathsf{N}_e+\mathsf{M}_e]^{-1}\boldsymbol{\cdot}\,\mathsf{M}_e\big]_{01}$, and $c_{01}\equiv[\mathsf{N}_e+\mathsf{M}_e]^{-1}_{01}/\,[\mathsf{N}_e+\mathsf{M}_e]^{-1}_{00} = -(\nu_{e01}+\mu_{e01})/(\nu_{e11}+\mu_{e11})$. Neglecting electron heat flow effects and $U_{i\theta}$, we obtain $b_{01}\to 0$ and the N_e and M_e matrices reduce to their 00 elements; then, the bootstrap current is given approximately by $\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{J}_{\rm bs}\rangle \simeq [\mu_{e00}/(\nu_{e00}+\mu_{e00})]\,[-I(dP_0/d\psi_0)] \sim -\sqrt{\epsilon}\,(B_0/B_p)(dP_0/d\rho)$ in the banana regime. The heat flow source effects in (30) are illustrative of the extra electron cyclotron current drive effects discussed in [12].

Next we determine $U_{i\theta}$ and $Q_{i\theta}$. The reference ion collision frequency is analogous to (20) with $e\to i$: $1/\tau_{ii} = (4/3\sqrt{\pi})(4\pi\, n_i Z_i^4 e^4 \ln\Lambda)/(\{4\pi\epsilon_0\}^2\, m_i^2\, v_{Ti}^3)$. The ion friction and viscosity coefficient matrices N_i and M_i are the same as the electron ones in (22) and (21) with [10] $Z_{\rm eff}\to Z_*$ in which for a combination of hydrogenic ions (subscript i, $Z_i=1$) and a small admixture of impurity ions (subscript I) $Z_*\equiv\sum_I n_I Z_I^2/n_i = (n_e/n_i)Z_{\rm eff}-1$. Assuming impurities have the same flow velocities as hydrogenic ions, there is no collisional friction between them: $\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{R}}_{i{\rm V}}\rangle = 0$. Expanding (3) as in (6), averaging over ζ, FSA and summing over species, the plasma equilibrium ($t\gg 1/\nu_i$) parallel force balance becomes [6, 7]

$$0 \simeq -\,\langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\,\boldsymbol{\pi}_{i\|}\rangle + \langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{\rm V}\rangle. \qquad (32)$$

Here, the summed parallel viscous force has been approximated by its ion component because it is larger than the electron one by $\sqrt{m_i/m_e} \gg 1$. Also, the externally-imposed and fluctuation-induced (due to Reynolds and Maxwell stresses) FSA parallel forces on the plasma are $\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{\rm V}\rangle = \langle \mathbf{B}_0\boldsymbol{\cdot}\sum_s(\overline{\mathbf{S}}_{s{\rm V}} - m_s\overline{\mathbf{V}}_s\overline{S}_{sn})\rangle - \sum_s\langle \mathbf{B}_0\boldsymbol{\cdot}(m_s n_{s0}\overline{\tilde{\mathbf{V}}_s\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_s} + \overline{\boldsymbol{\nabla\cdot}\boldsymbol{\pi}_{s\wedge}})\rangle + \langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\tilde{\mathbf{J}}_\wedge\times\tilde{\mathbf{B}}_\perp}\rangle$. As explained in the paragraph preceding (24), the fluctuating flows here are fluid flows that result from a combination of guiding center and diamagnetic flows.

To first order in the gyroradius expansion the FSA equilibrium parallel heat flow equation is

$$0 = -\,\langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\boldsymbol{\Theta}_{i\|}\rangle + \langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{R}}_{i{\rm q}}\rangle + (m_i/T_{i0})\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{i{\rm q}}\rangle. \qquad (33)$$

Here, we have $\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{i{\rm q}}\rangle \equiv \langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\ddagger}_{i{\rm q}}\rangle + (5/2)p_{i0}\overline{\tilde{\mathbf{V}}_i\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_i} - \langle \mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla\cdot}\left(\overline{\tilde{\mathbf{V}}_i\tilde{\mathbf{q}}_i + \tilde{\mathbf{q}}_i\tilde{\mathbf{V}}_i + (2/3)\tilde{\mathbf{V}}_i\boldsymbol{\cdot}\,\tilde{\mathbf{q}}_i\mathbf{I}}\right)\rangle$. Since the heat friction $\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{R}}_{i{\rm q}}\rangle = -\,(m_i n_{i0}/\tau_{ii})(-2/5 n_{i0}T_{i0})\langle \mathbf{B}_0\boldsymbol{\cdot}\mathbf{q}_i\rangle$, using (13) and the ion version of (18) this equation can be solved to yield

$$Q_{i\theta} = -\,c_U U_{i\theta} - \frac{c_T\, I}{q_i\langle B_0^2\rangle}\,\frac{dT_{i0}}{d\psi_p} + \frac{\tau_{ii}\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{i{\rm q}}\rangle/\langle B_0^2\rangle}{m_i n_{i0}(\nu_{i11}+\mu_{i11})}. \qquad (34)$$

Here, $c_U\equiv\mu_{i01}/(\nu_{i11}+\mu_{i11})$ and $c_T\equiv\nu_{i11}/(\nu_{i11}+\mu_{i11})$. Substituting this result into (32) yields

$$U_{i\theta} = k_i\,\frac{I}{q_i\langle B_0^2\rangle}\,\frac{dT_{i0}}{d\psi_p} + U^{\rm drives}_{i\theta}. \qquad (35)$$

The neoclassical offset poloidal flow coefficient is

$$k_i \equiv \frac{\mu_{i01}}{\mu_{i00}}\,\frac{1}{1+(\mu_{i11}-\mu_{i01}^2/\mu_{i00})/\nu_{i11}}. \qquad (36)$$

For a pure electron-ion plasma (i.e., $Z_*\to 0$) in the asymptotic ion banana collisionality regime ($\nu_{*i}\ll 1$, $\sqrt{\epsilon}\ll 1$), this yields the usual results: $\mu_{i01}/\mu_{i00} = 1.17$ and $k_i = 1.17/(1+0.67\sqrt{\epsilon})$. Possible poloidal flows driven by external sources and fluctuations are

$$U^{\rm drives}_{i\theta} = \frac{c_{\rm dr}}{m_i n_{i0}\langle B_0^2\rangle}\left(\frac{\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{\rm V}\rangle}{\mu_{i00}/\tau_{ii}} + \frac{(m_i/T_{i0})\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{S}}{}^{\rm tot}_{i{\rm q}}\rangle}{(\nu_{i11}+\mu_{i11})/\tau_{ii}}\right). \qquad (37)$$

Here, $c_{\rm dr} = (\nu_{i11}+\mu_{i11})/(\nu_{i11}+\mu_{i11}-\mu_{i01}^2/\mu_{i00})$.

Since $\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}\theta = U_{i\theta}\,\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\theta = (I/qR^2)\,U_{i\theta}$, substituting this poloidal ion flow into (8) yields the relation

$$\Omega_t = -\left(\frac{d\Phi}{d\psi_p} + \frac{1}{n_{i0}q_i}\frac{dp_{i0}}{d\psi_p}\right) + \Omega_{*{\rm p}}, \quad \Omega_{*{\rm p}} \equiv \frac{I}{R^2}U_{i\theta}. \qquad (38)$$

This relation does not determine either Ω_t or $d\Phi_0/d\psi_p$; it only yields a relation between the radial electric field ($\propto d\Phi_0/d\psi_p$) and the toroidal rotation frequency Ω_t. The toroidal rotation (or radial electric field) is determined from the non-ambipolar particle fluxes on the longer plasma transport time scale [7].

IV. MAGNETIC FLUX TRANSIENT EFFECTS

The poloidal and toroidal magnetic fluxes ψ_p and ψ_t evolve during plasma start-up, addition of current-drives, and approach to steady state on magnetic field diffusion time scales. These "slow" $\mathcal{O}\{\delta^2\}$ effects have been negligible in the preceding $\mathcal{O}\{\delta^0,\delta^1\}$ analyses, but need to be included in the comprehensive transport equations. Using $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ with $\mathbf{A}=\psi_t\boldsymbol{\nabla}\theta-\psi_p\boldsymbol{\nabla}\zeta$ in Faraday's law in the form $\boldsymbol{\nabla}\times(\partial\mathbf{A}/\partial t|_{\mathbf{x}}-\boldsymbol{\nabla}\phi+\mathbf{E}^A)=\mathbf{0}$ and the FSA of R^{-2} times these equations yields [7] for the toroidal and poloidal flux evolution equations:

$$\left.\frac{\partial\psi_t}{\partial t}\right|_{\mathbf{x}} \;=\; -\,\bar{u}_G\frac{\partial\psi_t}{\partial\rho} \equiv \dot{\psi}_t, \qquad (39)$$

$$\left.\frac{\partial\psi_p}{\partial t}\right|_{\mathbf{x}} \;=\; \frac{\langle \mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{E}}{}^A\rangle}{I\langle R^{-2}\rangle} - \frac{\partial\Psi}{\partial t} - \bar{u}_G\frac{\partial\psi_p}{\partial\rho}. \qquad (40)$$

Here, $\bar{u}_G \equiv \langle \mathbf{u}_G \cdot \boldsymbol{\nabla}\rho \rangle = \langle \mathbf{B}_p \cdot \overline{\mathbf{E}}^A \rangle / (\psi_p' I \langle R^{-2} \rangle)$ is the ψ_t "Grid speed" [13] and $2\pi\, \partial\Psi/\partial t \equiv V_{\rm loop}^{\zeta}(t)$ is the toroidal loop voltage induced by the ohmic heating solenoid. Using the parallel Ohm's law in (26) for $\langle \mathbf{B}_0 \cdot \overline{\mathbf{E}}^A \rangle$ and (15), (16) for the parallel current $\mu_0 \langle \mathbf{B}_0 \cdot \mathbf{J} \rangle$ yields a diffusion equation for poloidal flux ψ_p on a toroidal flux surface ψ_t:

$$\dot{\psi}_p \;\equiv\; \frac{\partial \psi_p}{\partial t}\bigg|_{\psi_t} = D_\eta\, \Delta^+ \psi_p - S_\psi, \tag{41}$$

$$D_\eta \;\equiv\; \frac{\eta_\parallel^{\rm nc}}{\mu_0}, \quad S_\psi = \frac{\partial \Psi}{\partial t} + \frac{\eta_\parallel^{\rm nc}}{I \langle R^{-2} \rangle} \langle \mathbf{B}_0 \cdot \mathbf{J}_{\rm drives} \rangle. \tag{42}$$

Tokamak plasma properties are determined in terms of the poloidal magnetic flux ψ_p: 1) the Grad-Shafranov equation determines $\psi_p(\mathbf{x})$ given $P(\psi_p)$ and $I(\psi_p)$; 2) classical and neoclassical transport are determined [14] across poloidal flux surfaces ψ_p; and 3) the drift-kinetic and gyrokinetic equations use poloidal flux variables and $f_0 = f_{\rm Max}(\psi_p)$ — so the canonical toroidal angular momentum emerges as a natural constant of motion. Thus, we need to transform [14] the fluid moment equations from determining the species density, momentum, energy at a laboratory position $\mathbf{x}$ to determining them on a poloidal flux surface ψ_p — i.e., $\partial n/\partial t|_{\mathbf{x}} \implies \partial n/\partial t|_{\psi_p}$ etc. However, for low collisionality tokamak plasmas this transformation should first be made [14] in the drift-kinetic (or gyrokinetic) equation. This transformation adds [15] a paleoclassical diffusion-type operator $\mathcal{D}\{f\} \sim D_\eta f/a^2 \sim \mathcal{O}\{\delta^2\}$ to the right side of (1).

The effects of the paleoclassical transport operator $\mathcal{D}$ will be illustrated through their influence on the perturbed, ζ-averaged and FSA density equation (2). First, we note that [7, 15]

$$\langle \mathcal{D}\{n_0\} \rangle \equiv -\,\dot{\rho}_{\psi_p} \frac{\partial n_0}{\partial \rho} + \langle \boldsymbol{\nabla} \cdot n_0 \mathbf{u}_G \rangle + \frac{1}{V'} \frac{\partial^2}{\partial \rho^2} (V' \bar{D}_\eta n_0). \tag{43}$$

Here, $\dot{\rho}_{\psi_p} \equiv \dot{\psi}_p/\psi_p'$ with $\psi_p' \equiv d\psi_p/d\rho$ represents ψ_p surface motion relative to the ψ_t-based radial coordinate ρ, $\bar{D}_\eta \equiv D_\eta/\bar{a}^2$, $1/\bar{a}^2 \equiv \langle |\boldsymbol{\nabla}\rho|^2/R^2 \rangle / \langle R^{-2} \rangle \simeq 1/a^2$ and $\langle \boldsymbol{\nabla} \cdot \mathbf{u}_G \rangle = (1/V')(\partial V'/\partial t|_\rho$. Including these transformation effects, the FSA density equation becomes [7]

$$\frac{1}{V'} \frac{\partial}{\partial t}\bigg|_{\psi_p} (V' n_0) + \dot{\rho}_{\psi_p} \frac{\partial n_0}{\partial \rho} + \frac{1}{V'} \frac{\partial}{\partial \rho} (V' \, \Gamma) = \langle \overline{S}_n \rangle. \tag{44}$$

The quantity $V' n_0$ is the number of particles between the ρ and $\rho + d\rho$ surfaces; it is an adiabatic plasma property. The radial coordinate $\rho \equiv \sqrt{\psi_t/\pi B_{t0}}$ in transport codes [4, 5] is based on the relatively immobile [2, 3, 13] toroidal magnetic flux ψ_t; however, the fluid moments $n, T, \mathbf{V}$ are determined on poloidal flux surfaces ψ_p. The $\dot{\rho}_{\psi_p} \partial n_0/\partial \rho$ term takes account of the slow (magnetic field diffusion time scale) motion of the ψ_p surfaces relative to the ψ_t surfaces. The total $\mathcal{O}\{\delta^2\}$ particle flux for each species is [7]:

$$\Gamma \equiv \langle \boldsymbol{\Gamma} \cdot \boldsymbol{\nabla} \rho \rangle = \Gamma^a + \Gamma^{na} + \Gamma^a_{\rm pc}. \tag{45}$$

Here, $\Gamma^a \equiv n_0 \langle (\overline{\mathbf{V}}_2 - \mathbf{u}_G) \cdot \boldsymbol{\nabla} \rho \rangle$ is the intrinsically ambipolar particle flux driven by axisymmetric collisional processes, $\Gamma^{na} \equiv \langle \overline{\tilde{n}_1 \tilde{\mathbf{V}}_1} \cdot \boldsymbol{\nabla} \rho \rangle$ is the possibly non-ambipolar fluctuation-induced particle flux and $\Gamma^a_{\rm pc} \equiv (\partial/\partial\rho)(V' \bar{D}_\eta n_0)$ is the ambipolar paleoclassical particle flux that results from the coordinate transformation [7].

V. RADIAL PARTICLE FLUXES, PLASMA TOROIDAL ROTATION AND ELECTRIC FIELD

In sections II and III the $\mathcal{O}\{\delta^0\}$ radial and $\mathcal{O}\{\delta^1\}$ parallel components of the species force balance equations (3) were analyzed. The $\mathcal{O}\{\delta^2\}$ particle fluxes will be determined from the toroidal angular component of the force balance equations. A vector identity for determining the second order radial fluxes is ($\mathbf{e}_\zeta \equiv R^2 \boldsymbol{\nabla} \zeta = R\, \hat{\mathbf{e}}_\zeta$)

$$\mathbf{e}_\zeta \cdot n \mathbf{V} \times \mathbf{B}_0 = -\, n \mathbf{V} \cdot \mathbf{e}_\zeta \times \mathbf{B}_0 = (n \mathbf{V} \cdot \boldsymbol{\nabla} \rho)\, \psi_p'. \tag{46}$$

Thus, the $\mathbf{e}_\zeta$ component of the force balance shows the particle flux is induced by j toroidal torques $T_{\zeta j} \equiv \mathbf{e}_\zeta \cdot \mathbf{F}_j$ on the plasma species by the various fluid forces $\mathbf{F}_j$:

$$0 = \mathbf{e}_\zeta \cdot (nq \mathbf{V} \times \mathbf{B}_0 + \sum_j \mathbf{F}_j) \implies q \psi_p' \Gamma = -\sum_j T_{\zeta j}. \tag{47}$$

Taking the toroidal angular ($\mathbf{e}_\zeta \cdot$) component of the species force balance, transforming from $\mathbf{x}$ to ψ_p, and averaging over ζ and a flux surface yields a particle flux with 15 components [7]. There are 6 collision-induced intrinsically ambipolar fluxes due to toroidal torques from cross friction (classical), parallel friction (Pfirsch-Schlüter), parallel viscosity (banana-plateau), non-inductive current drives, dynamos and the $\overline{\mathbf{E}}^A \times \mathbf{B}_p$ pinch, plus the paleoclassical one due to the coordinate transformation. In addition, there are 8 possibly non-ambipolar fluxes due to toroidal torques exerted on plasma species by polarization flows induced by $\partial \Omega_t/\partial t \neq 0$, neoclassical toroidal viscosity (NTV) from small 3D NA magnetic fields, perpendicular viscosities (classical, neoclassical and paleoclassical), Reynolds and Maxwell stresses induced by fluctuations, $\mathbf{J} \times \mathbf{B}$ forces in the vicinity of low order rational surfaces induced by resonant field errors, poloidal flux transients, and external momentum sources [7].

For quasineutrality the transport time scale charge continuity equation requires $\langle \mathbf{J} \cdot \boldsymbol{\nabla} \rho \rangle = 0$. Setting $\langle \mathbf{J} \cdot \boldsymbol{\nabla} \rho \rangle$ to zero yields [6, 7] a comprehensive toroidal torque balance equation for the toroidal angular momentum density $L_t \equiv m_i n_{i0} \langle R^2 \Omega_t \rangle$ (neglecting small electron terms):

$$\begin{aligned} \frac{1}{V'} \frac{\partial}{\partial t}\bigg|_{\psi_p} (V' L_t) \;\simeq\; & -\langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \overline{\boldsymbol{\pi}}_{i\parallel}^{\rm NA} \rangle - \langle \mathbf{e}_\zeta \cdot \boldsymbol{\nabla} \cdot \overline{\boldsymbol{\pi}}_{i\perp} \rangle \\ & - \frac{1}{V'} \frac{\partial}{\partial \rho} (V' \Pi_{i\rho\zeta}) + \langle \mathbf{e}_\zeta \cdot \overline{\tilde{\mathbf{J}} \times \tilde{\mathbf{B}}} \rangle \\ & - \dot{\rho}_{\psi_p} \frac{\partial L_t}{\partial \rho} + \langle \mathbf{e}_\zeta \cdot {\textstyle\sum_s} \bar{\mathbf{S}}_{sm} \rangle. \end{aligned} \tag{48}$$

Here, $V' L_t$ is the toroidal angular momentum between the ρ and $\rho + d\rho$ flux surfaces, which is an adiabatic quantity. The terms on the right represent NTV effects, collision-induced perpendicular viscosity, Reynolds and Maxwell stresses induced by fluctuations, poloidal flux transients and externally supplied momentum sources.

Once this equation is solved for L_t the toroidal rotation is given by $\langle\Omega_t\rangle \equiv L_t/(m_i n_{i0}\langle R^2\rangle)$. Hence from (38) the radial electric field $E_\rho \equiv -\,|\boldsymbol{\nabla}\rho\,|\,(d\Phi_0/d\rho)$ is [7]:

$$E_\rho = |\boldsymbol{\nabla}\rho\,|\left[\left(\langle\Omega_t\rangle - \frac{\langle R^2\Omega_{*p}\rangle}{\langle R^2\rangle}\right)\psi_p' + \frac{1}{n_{i0}q_i}\,\frac{dp_{i0}}{d\rho}\right]. \quad (49)$$

This E_ρ (or Ω_t) causes the electron and ion non-ambipolar radial particle fluxes to become equal (i.e., ambipolar): $\Gamma_e^{na}(E_\rho) = Z_i\,\Gamma_i^{na}(E_\rho)$ — to satisfy $\langle\mathbf{J}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle = 0$, which was used to obtain the Ω_t (or E_ρ) equation. Hence, the net ambipolar particle flux is the sum of the intrinsically ambipolar fluxes $\Gamma^a + \Gamma^a_{\rm pc}$ and the potentially non-ambipolar fluxes evaluated at the electric field E_ρ, $\Gamma^{na}(E_\rho)$. It is easiest to evaluate for electrons since the dominant (nearly canceling) terms in the $\langle\mathbf{J}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle$ that is set to zero to obtain (48) result from non-ambipolar ion particle fluxes (this is the so-called "ion root" [16]):

$$\Gamma_e^{\rm net} \equiv \Gamma_e^a + \Gamma^a_{e{\rm pc}} + \Gamma_e^{na}(E_\rho) = \Gamma_i^{\rm net}. \quad (50)$$

This is the net ambipolar particle flux Γ to be used in the density continuity equation on a ψ_p flux surface given in (44). See Eqs. (87)–(93) in Ref. [7] for the 7 intrinsically ambipolar particle fluxes and Eqs. (106)–(113) there for the 8 potentially non-ambipolar particle fluxes.

VI. ENERGY TRANSPORT EQUATIONS

The energy transport equations on the transport time scale ($\partial/\partial t \sim \delta^2$) are obtained by expanding all quantities in (4) using perturbation expansions like that in (6). Then, we take the toroidal angle (ζ) average of the equations. Finally, we flux surface average the resultant equations and transform them from laboratory ($\mathbf{x}$) to poloidal flux (ψ_p) coordinates to obtain for each species

$$\frac{3}{2}\,p_0\,\frac{\partial}{\partial t}\bigg|_{\psi_p}\ln\,(p_0 V'^{5/3}) + \frac{3}{2}\dot{\rho}_{\psi_p}\frac{\partial p_0}{\partial t} + \frac{1}{V'}\,\frac{\partial}{\partial\rho}\,(V'\Upsilon) = Q_{\rm net}. \quad (51)$$

Here, $\ln\,(p_{s0}V'^{5/3})$ is the collisional entropy between the ρ and $\rho + d\rho$ flux surfaces; without dissipation or energy sources it is concerved. The $\dot{\rho}_{\psi_p}$ term accounts for motion of ψ_p surfaces relative to the ψ_t surfaces.

The total FSA "radial" heat fluxes are

$$\begin{aligned}\Upsilon \;&\equiv\; \left\langle\left(\overline{\mathbf{q}}_2 + \frac{5}{2}\,[\,p_0(\overline{\mathbf{V}}_2 - \mathbf{u}_G) + \overline{\tilde{p}_1\tilde{\mathbf{V}}_1}\,]\right)\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\right\rangle + \Upsilon_{\rm pc}\\ &= \left\langle\left(\overline{\mathbf{q}}_2 + \frac{5}{2}\,n_0\overline{\tilde{T}_1\tilde{\mathbf{V}}_1}\right)\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\right\rangle + \Upsilon_{\rm pc} + \frac{5}{2}T_0\Gamma, \quad (52)\end{aligned}$$

which is comprised of conductive plus convective heat fluxes. The particle flux Γ in the convective heat flux $(5/2)T_0\Gamma$ is the net ambipolar one specified in (50). The second order conductive heat flux $\overline{\mathbf{q}}_2$ is obtained similarly to how the radial particle fluxes were obtained [7] via (46) from the toroidal angular component of the perturbed, ζ-averaged, FSA and transformed heat flow equation (5):

$$\langle\overline{\mathbf{q}}_2\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle = \Upsilon_{\rm cl} + \Upsilon_{\rm PS} + \Upsilon_{\rm bp} + \Upsilon_{\rm fl} + \Upsilon_{\rm Sq}. \quad (53)$$

The lowest order classical (cross heat friction), Pfirsch-Schlüter (parallel heat friction), banana-plateau (parallel heat viscosity) and paleoclassical (transform to ψ_p) collision-induced heat fluxes, and fluctuation- and source-induced conductive radial heat fluxes obtained from the $\mathbf{e}_\zeta \equiv R^2\boldsymbol{\nabla}\zeta$ component of (5) for each species are

$$\Upsilon_{\rm cl} \;\equiv\; T_{s0}\left\langle\frac{\mathbf{B}_0\times\boldsymbol{\nabla}\rho}{q_s B_0^2}\boldsymbol{\cdot}\overline{\mathbf{R}}_{\rm sq}\right\rangle, \quad (54)$$

$$\Upsilon_{\rm PS} \;\equiv\; -\frac{I\,T_{s0}}{q_s\psi_p'}\left\langle\left(\frac{1}{B_0^2} - \frac{1}{\langle B_0^2\rangle}\right)\mathbf{B}_0\boldsymbol{\cdot}\overline{\mathbf{R}}_{s{\rm q}}\right\rangle, \quad (55)$$

$$\Upsilon_{\rm bp} \;\equiv\; \frac{I\,T_{s0}}{q_s\langle B_0^2\rangle\psi_p'}\,\langle\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{\cdot}\,\overline{\boldsymbol{\Theta}}_{s\|}\rangle, \quad (56)$$

$$\Upsilon_{\rm pc} \;\equiv\; -\frac{1}{V'}\,\frac{\partial}{\partial\rho}\left(V'\bar{D}_\eta\,\frac{3}{2}n_{s0}T_{s0}\right), \quad (57)$$

$$\begin{aligned}\Upsilon_{\rm fl} \;\equiv\; &\frac{m_s}{q_s\psi_p'}\left[\frac{5}{2}\left(\frac{\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\,\overline{\tilde{p}_{s1}\boldsymbol{\nabla}\tilde{T}_{s1}}\,\rangle}{m_s} + p_{s0}\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\overline{\tilde{\mathbf{V}}_{s1}\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{\mathbf{V}}_{s1}}\,\rangle\right)\right.\\ &\left.+\,\frac{1}{V'}\frac{\partial}{\partial\rho}\Big(V'\langle\mathbf{e}_\zeta\boldsymbol{\cdot}(\overline{\tilde{\mathbf{V}}_{s1}\tilde{\mathbf{q}}_{s1} + \tilde{\mathbf{q}}_{s1}\tilde{\mathbf{V}}_{s1} + (2/3)\tilde{\mathbf{V}}_{s1}\boldsymbol{\cdot}\tilde{\mathbf{q}}_{s1}\mathbf{I}}\,)\boldsymbol{\cdot}\boldsymbol{\nabla}\rho\rangle\Big)\right]\\ &-\frac{1}{\psi_p'}\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\,\overline{\tilde{\mathbf{q}}_{s1}\times\tilde{\mathbf{B}}}\,\rangle, \quad (58)\end{aligned}$$

$$\Upsilon_{\rm Sq} \;\equiv\; -\frac{m_s}{q_s\psi_p'}\langle\mathbf{e}_\zeta\boldsymbol{\cdot}\,\overline{\mathbf{S}}^{\ddagger}_{s{\rm q}}\rangle. \quad (59)$$

Here, the perpendicular viscosity effects [7] have been neglected since they give $\mathcal{O}\{\delta^2\}$ smaller contributions to the heat fluxes. The rather complicated fluctuation-driven contributions in $\Upsilon_{\rm fl}$ given by (58) are usually much smaller than the typically dominant $(5/2)n_0\langle\,\overline{\tilde{T}_1\tilde{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\rho}\rangle$ fluctuation-driven contribution to (52) — because they mostly involve the smaller flow fluctuation magnitudes in the toroidal direction compared to the typically larger ones in the radial direction. As explained in the paragraph preceding (24), the fluctuating flows and heat flows here are fluid flows that result from a combination of guiding center and diamagnetic flows. However, as with the fluctuation-induced particle flux (see the discussion in Appendix A of Ref. [7]), the fluctuating diamagnetic flow does not contribute to the fluctuation-induced heat transport; thus the lowest order Υ is just due to guiding center flow, i.e., $(5/2)n_0\langle\,\overline{\tilde{T}_1\tilde{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\rho}\rangle \simeq (5/2)n_0\langle\,\overline{\tilde{T}_1\tilde{\mathbf{V}}_g\boldsymbol{\cdot}\boldsymbol{\nabla}\rho}\rangle$.

Finally, the net rate of energy input to the species $Q_{\rm net}$ in (51) is given in general by

$$\begin{aligned}Q_{\rm net} \;\equiv\; &\langle\,\overline{Q}\,\rangle + \langle(\overline{\mathbf{V}}_2 - \mathbf{u}_G)\boldsymbol{\cdot}\boldsymbol{\nabla}p_0\rangle + \langle\,\overline{\tilde{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\tilde{p}_1}\,\rangle\\ &+\langle\,\overline{\mathbf{V}}_1\boldsymbol{\cdot}\boldsymbol{\nabla}\overline{p}_1\,\rangle - \langle\overline{\boldsymbol{\pi}\,\boldsymbol{:}\,\boldsymbol{\nabla}\mathbf{V}_1}\rangle + \langle\overline{S}^{\ddagger}_E\rangle. \quad (60)\end{aligned}$$

Successive terms on the right will be specified in greater detail in the following paragraphs.

The total collisional energy exchange between species is [1] $Q \equiv \int d^3v\,(m|\mathbf{v}-\mathbf{V}|^2/2)\,\mathcal{C}\{f\} = \int d^3v\,(mv^2/2 - m\mathbf{v}\cdot\mathbf{V})\,\mathcal{C}\{f\} = \text{sign}\{q_s\}Q_\Delta - \mathbf{V}\cdot\mathbf{R}_\text{V}$. Thus, the FSA of the ζ-average of Q has three parts:

$$\langle\,\overline{Q}\,\rangle \equiv \text{sign}\{q_s\}\langle\overline{Q}_\Delta\rangle - \langle\,\overline{\mathbf{V}}_1\cdot\overline{\mathbf{R}}_\text{V}\,\rangle - \langle\,\overline{\tilde{\mathbf{V}}_1\cdot\tilde{\mathbf{R}}_\text{V}}\,\rangle. \quad (61)$$

The first term is the collisional energy transfer rate from electrons to ions due to their differing temperatures. For ions it is $Q_\Delta \equiv -\int d^3v\,(m_e v^2/2)\,\mathcal{C}_{ei}\{f_e\} = 3\,(m_e/m_i)(n_e/\tau_{ee})(T_e - T_i)$; it is $-\,Q_\Delta$ in the electron energy equation, because of collisional energy conservation. As discussed in the next paragraph, the second term represents primarily Joule heating. The final term in (61) represents fluctuation-induced parallel and perpendicular Joule heating; it has been found to be negligible for some typical tokamak plasma conditions [17].

The net collisional Joule heating rate can be obtained by neglecting the last term in (61) and summing over species: $\langle\,\overline{Q}_\text{J}\,\rangle \equiv \sum_s \langle\,\overline{Q}_s\,\rangle = -\sum_s \langle\,\overline{\mathbf{V}}_{s1}\cdot\overline{\mathbf{R}}_{s\text{V}}\,\rangle$. The first order flows and currents can be represented in terms of their parallel and cross components by [2, 3, 7]

$$\mathbf{V}_{s1} = U_{s\theta}(\psi_p)\mathbf{B}_0 + \Omega_{s\wedge}(\psi_p)\mathbf{e}_\zeta, \quad (62)$$

$$\Omega_{s\wedge} \equiv -\left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{s0}q_s}\frac{dp_{s0}}{d\psi_p}\right), \quad (63)$$

$$\mathbf{J} \equiv \sum_s n_{s0}q_s\overline{\mathbf{V}}_{s1} = K_J(\psi_p)\mathbf{B}_0 - \frac{dP_0(\psi_p)}{d\psi_p}\mathbf{e}_\zeta. \quad (64)$$

As noted before (14), $\langle B_0^2\rangle K_J = \langle\mathbf{B}_0\cdot\mathbf{J}\rangle + I\,dP_0/d\psi_p$. Also, we find from Ampere's law $\mu_0\mathbf{J} \equiv \boldsymbol{\nabla}\times\mathbf{B}_0 = \boldsymbol{\nabla}I\times\boldsymbol{\nabla}\zeta + \Delta^*\psi_p\boldsymbol{\nabla}\zeta$, that $K_J \equiv \mathbf{J}\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta = -\,(1/\mu_0)\,dI/d\psi_p$. Thus, the toroidal and poloidal components of the current are ($\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta\rangle = I\langle R^{-2}\rangle$)

$$\frac{\langle\mathbf{J}\cdot\boldsymbol{\nabla}\zeta\rangle}{\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta\rangle} = \frac{\langle\mathbf{B}_0\cdot\mathbf{J}\rangle}{\langle B_0^2\rangle} - \frac{dP_0/d\psi_p}{\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta\rangle}\left(1 - \frac{I^2\langle R^{-2}\rangle}{\langle B_0^2\rangle}\right), \quad (65)$$

$$\frac{\mathbf{J}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta} \equiv K_J(\psi_p) = -\frac{1}{\mu_0}\frac{dI}{d\psi_p}. \quad (66)$$

Note that the toroidal current $\propto \langle\mathbf{J}\cdot\boldsymbol{\nabla}\zeta\rangle$ has both a FSA component and a (small) Pfirsh-Schlüter type component. The poloidal current $\propto \langle\mathbf{J}\cdot\boldsymbol{\nabla}\theta\rangle$ is typically smaller than the toroidal current by a factor of $B_p/B_t \sim \epsilon/q$.

Using these flow and current representations, the first row of (17) for the FSA parallel electron friction force $\langle\mathbf{B}_0\cdot\overline{\mathbf{R}}_{e\text{V}}\rangle$, the relation $\langle\mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle = \langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta\rangle\,(\langle\mathbf{e}_\zeta\cdot\overline{\mathbf{E}}^A\rangle + \langle\mathbf{u}_G\cdot\boldsymbol{\nabla}\psi_p\rangle)$ and from collisional momentum conservation $\mathbf{R}_{i\text{V}} = -\,\mathbf{R}_{e\text{V}}$, we find for the Joule heating

$$\begin{aligned}\langle\,\overline{Q}_\text{J}\,\rangle &\equiv -\sum_s \left(U_{s\theta}\langle\,\mathbf{B}_0\cdot\overline{\mathbf{R}}_{s\text{V}}\,\rangle + \Omega_{s\wedge}\langle\,\mathbf{e}_\zeta\cdot\overline{\mathbf{R}}_{s\text{V}}\,\rangle\right)\\ &= K_J\frac{\langle\mathbf{B}_0\cdot\overline{\mathbf{R}}_{e\text{V}}\rangle}{n_{e0}e} - \frac{\Gamma_e^a}{n_{e0}}\frac{dP_0}{d\rho} - \frac{\langle\mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle}{\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta\rangle}\frac{dP_0}{d\psi_p}\\ &= \langle\mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle\frac{\langle\mathbf{J}\cdot\boldsymbol{\nabla}\zeta\rangle}{\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta\rangle} - \frac{\Gamma_e^a}{n_{e0}}\frac{dP_0}{d\rho}\\ &\quad + \frac{\mathbf{J}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta}\left(\frac{\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}}_{e\|}\rangle - \langle\mathbf{B}_0\cdot\overline{\mathbf{S}}_{e\text{V}}^\text{tot}\rangle}{n_{e0}e}\right). \quad (67)\end{aligned}$$

The $P_0\,dV$ work by the collision-induced classical, Pfirsch-Schlüter and banana-plateau particle flux Γ_e^a is usually negligible. For ohmically heated tokamak plasmas where the Joule heating from the average toroidal current induced by the toroidal inductive electric field usually dominates over that induced by the poloidal current (for $B_p/B_t \sim \epsilon/q \ll 1$), we obtain simply $\langle Q_J\rangle \simeq \langle\mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle\,\langle\mathbf{B}_0\cdot\mathbf{J}\rangle/\langle B_0^2\rangle$. The FSA inductive parallel electric field $\langle\mathbf{B}_0\cdot\overline{\mathbf{E}}^A\rangle$ is provided by the extended parallel neoclassical Ohm's law given in (26).

Implicitly, the Joule heating result in (67) is obtained in the ion rest frame. Thus, for ions we have

$$\langle\,\overline{Q}_i\,\rangle \equiv \langle\,\overline{Q}_\Delta\rangle - \langle\,\overline{\tilde{\mathbf{V}}_{i1}\cdot\tilde{\mathbf{R}}_{i\text{V}}}\,\rangle. \quad (68)$$

The corresponding collisional energy exchange rate for electrons is

$$\langle\,\overline{Q}_e\,\rangle \equiv \langle\,\overline{Q}_J\,\rangle - \langle\,\overline{Q}_\Delta\rangle - \langle\,\overline{\tilde{\mathbf{V}}_{e1}\cdot\tilde{\mathbf{R}}_{e\text{V}}}\,\rangle. \quad (69)$$

The second Q_net term in (60) represents the $p\,dV$ work done by an outflowing species of particles. For either species it is

$$\langle(\overline{\mathbf{V}}_{s2} - \mathbf{u}_G)\cdot\boldsymbol{\nabla}p_{s0}\rangle = (\Gamma_e^a/n_{s0})(dp_{s0}/d\rho). \quad (70)$$

This contribution cancels part of the small $P_0\,dV$ work term in (67). The third Q_net term in (60) represents $p\,dV$ work by fluctuations as they induce radial particle transport. Using the vector identity $\mathbf{V}\cdot\boldsymbol{\nabla}p = \boldsymbol{\nabla}\cdot p\mathbf{V} - p\boldsymbol{\nabla}\cdot\mathbf{V}$, it can be seen that when fluctuating flows are incompressible the remaining $\langle\boldsymbol{\nabla}\cdot\overline{\tilde{p}_1\tilde{\mathbf{V}}_1}\rangle = (1/V')(\partial/\partial\rho)(V'\langle\overline{\tilde{p}_1\tilde{\mathbf{V}}_1}\cdot\boldsymbol{\nabla}\rho\rangle)$ term just changes the 5/2 factor in the convective heat flow in (52) to 3/2, as is well known. The fourth and fifth Q_net terms in (60) represent viscous heating. Using the vector identity that $\boldsymbol{\pi}:\boldsymbol{\nabla}\mathbf{V} = \boldsymbol{\nabla}\cdot(\mathbf{V}\cdot\boldsymbol{\pi}) - \mathbf{V}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$, the flow velocity representation in (62) and the fact that [2, 3, 7] $\langle\mathbf{e}_\zeta\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}}_\|^\text{A}\rangle = 0$, these viscous heating terms reduce for each species to [3]

$$Q_\text{visc} \equiv \langle\overline{\mathbf{V}}_1\cdot\boldsymbol{\nabla}\overline{p}_1\rangle - \langle\overline{\boldsymbol{\pi}}_\|:\boldsymbol{\nabla}\overline{\mathbf{V}}_1\rangle = U_\theta\,\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\overline{\boldsymbol{\pi}}_\|\rangle, \quad (71)$$

which is $\sim mn\mu U_\theta^2\langle B_0^2\rangle$. However, it just cancels part of the typically small poloidal current contribution to (67).

For electrons there is one additional term to be added to the left side of (51). It is the contribution due to helically resonant paleoclassical transport processes [18]:

$$\langle\boldsymbol{\nabla}\cdot\mathbf{q}_{e*}^\text{pc}\rangle = -\frac{M}{V'}\frac{\partial^2}{\partial\rho^2}\left(V'\bar{D}_\eta\frac{3}{2}p_{e0}\right) + \frac{3}{2}\dot{\rho}_{\psi_*}\frac{\partial p_{e0}}{\partial\rho}. \quad (72)$$

Here, $M \sim 10$ is a factor reflecting the distance, relative to the poloidal periodicity length, over which the electron temperature is equilibrated as it diffuses radially with the diffusing poloidal magnetic flux [18]. Also, $\dot{\rho}_{\psi_*} = -\,q\,(\partial\dot{\psi}_p/\partial\rho)/q'\psi_p'$ with $q' \equiv \partial q/\partial\rho$ reflects the ρ motion required to stay on the same $q \equiv (\partial\psi_t/\partial\rho)/(\partial\psi_p/\partial\rho)$ surface in transient poloidal flux situations where $\psi_p(\rho,t) \simeq \psi_p(\rho,0) + \dot{\psi}_p(\rho)t$. There is no paleoclassical M contribution to the ion energy balance because [18] the ion toroidal precessional drift frequency exceeds the ion collision frequency in most tokamak plasmas.

VII. SUMMARY

Comprehensive plasma flow and transport equations for describing tokamak plasmas have been derived with a kinetic-based approach using a small gyroradius expansion. Relevant fluid moment equations that include neoclassical-based kinetic closures and non-axisymmeric perturbations (externally imposed or from plasma fluctuations) have been averaged over the toroidal angle ζ and then flux surface averaged (FSA). The zeroth order Alfvén time scale radial force balance provides a constraint relation for the toroidal plasma rotation Ω_t in (8) and (38). The first order parallel force balances yield the neoclassical-type parallel Ohm's law in (26) and poloidal ion flow in (35) for times longer than collision times. The radial fluxes of particles Γ in (45) and heat Υ in (52)–(59) are obtained from toroidal angular components of the momentum and heat flow equations, respectively. The resultant tokamak plasma model includes transport time scale evolution equations for the toroidal magnetic flux ψ_t in (39), poloidal magnetic flux ψ_p in (41) and toroidal angular momentum density $L_t \equiv m_i n_{i0} \langle R^2 \Omega_t \rangle$ in (48), in addition to expanded versions of the FSA equations for the density n in (44) and species pressure $p \equiv nT$ in (51). The FSA toroidal rotation frequency $\langle \Omega_t \rangle \equiv L_t/(m_i n_{i0} \langle R^2 \rangle)$ determines the radial electric field E_ρ in (49) that is required to obtain the net ambipolar radial particle flux $\Gamma^{\rm net}$ in (50).

Key attributes of and consequences from this new approach for tokamak plasma transport equations are: 1) The derivation of the radial particle flux and toroidal flow are naturally joined. 2) The radial electric field is determined self-consistently and enforces ambipolar radial particle transport. 3) The "mean-field" effects of microturbulence-induced fluctuations on all plasma transport properties are included self-consistently — parallel Ohm's law (31), poloidal ion flow (37), particle fluxes (45), toroidal rotation (48), heat fluxes (52), (58) and heating (60), (61). 4) Source effects (e.g., energetic neutral beam momentum input and non-inductive current drives) are also included self-consistently. 5) Paleoclassical n, Ω_t and T diffusion and pinch effects plus the electron heat transport in (72) are included naturally. Finally, 6) the radial motion of n, Ω_t and T induced by $\dot{\psi}_p \neq 0$ poloidal flux transients are included for the first time. These plasma transport equations follow naturally from extended two-fluid moment equations; hence they are consistent with extended MHD code frameworks and could provide a basis for the Fusion Simulation Program.

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