UW CPTC Report 08-8: Low Collisionality Neoclassical Toroidal Viscosity in Tokamaks and Quasi-symmetric Stellarators Using an Integral-truncation Technique

A. J. Cole, C. C. Hegna, and J. D. Callen
University of Wisconsin, Madison, WI 53706-1609
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In this report, radial particle fluxes induced by non-axisymmetric magnetic perturbations for the low-collisionality "ν" and "$1/\nu$" regimes previously calculated by K.C. Shaing [Phys. Plasmas **10**, 1443 (2003)] are unified into a single particle flux (or toroidal viscous force). Provided pitch-angle scattering dominates over collisional energy exchange, the energy component of phase space can be decoupled into independent regions [$E > E_c$ for ν regime, $E < E_c$ for $1/\nu$ regime, with E_c determined by $\nu_i(E_c) = \epsilon\,\omega_E$] within which the perturbed distribution function can be calculated as in Shaing's work. Using a technique first employed in axisymmetric neoclassical theory [K.T. Tsang and J.D. Callen, Phys. Fluids **19**, 667 (1976)], the smoothed particle flux is constructed by summing the partial contributions from ν and $1/\nu$ banana drift effects, respectively. The complete NTV force is expressed in terms of the equilibrium flows and a temperature-gradient-determined "intrinsic" flow.

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I. INTRODUCTION

Static non-axisymmetric magnetic perturbations can affect plasma rotation in toroidally confined plasmas either by inducing resonant localized electromagnetic torques on rational surfaces, or through their modification of $|\vec{B}|$. In this latter case, variations along a field line induce non-ambipolar radial particle transport and produce a global neoclassical toroidal viscous force [NTV][1] which tries to rotate the plasma at a diamagnetic-like rotation rate. Rotation in toroidal plasmas benefits confinement and stability by shielding resonant magnetic perturbations, stabilizing resistive wall modes, and reducing transport. For these reasons, NTV has generated considerable experimental interest,[2–5] as it promises to provide an external input for toroidal rotation without inducing locked modes. Such capability would be a great benefit to ITER, since its present benchmark scenario[6] relies on an ohmic startup with an anticipated low toroidal rotation rate ($\sim$ 0.5 kHz), compared to present tokamaks.

The connection between radial particle transport and toroidal forces is made by taking the toroidal component of the momentum equation for species j and averaging over a flux surface, to find[7]

$$m_j n_j \frac{\partial}{\partial t}\left\langle \vec{V}_j \cdot \vec{e}_\zeta \right\rangle = Z_j e \Gamma_j - \left\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{\parallel j} \right\rangle + \ldots, \quad (1)$$

in which a number of effects including intrinsically ambipolar transport processes are not explicitly listed. The total species' particle flux is defined by $\Gamma_j \equiv \left\langle n_j \vec{V}_j \cdot \vec{\nabla}\Psi \right\rangle = \left\langle n_j \vec{e}_\zeta \cdot \vec{V}_j \times \vec{B} \right\rangle$, with Ψ a generalized radial magnetic flux variable, and $\vec{e}_\zeta$ the covariant toroidal base vector, both defined in section II. Each toroidal force in (1) may be identified with a particle flux; for example, the polarization flux is defined by $\Gamma_p = -m_j n_j \left\langle \frac{\partial}{\partial t}\vec{V}_j \cdot \vec{e}_\zeta \right\rangle / (Z_j e)$, while the non-ambipolar flux (the focus of this paper) is identified with the toroidal viscous stress via the "flux-friction" relation $\Gamma_j^{\rm na} = \left\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{\parallel j} \right\rangle / (Z_j e)$.[8] Thus, the toroidal momentum equation (1) is equivalent to the statement that the total radial particle flux for species j is a result of all the individual force-induced fluxes: $\Gamma_j = \Gamma_p + \Gamma_{\rm na} + \ldots$, etc. Summing (1) over species yields[7]

$$\rho_m \frac{\partial}{\partial t}\left\langle \vec{u}_j \cdot \vec{e}_\zeta \right\rangle = \left\langle \vec{J} \cdot \vec{\nabla}\Psi \right\rangle - \sum_j \left\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{\parallel j} \right\rangle + \ldots, \quad (2)$$

where $\rho_m \equiv \sum_j m_j n_j$, and $\rho_m \vec{u} \equiv \sum_j m_j n_j \vec{V}_j$ is the center of mass velocity. The flux-surface-averaged radial current, $\left\langle \vec{J} \cdot \vec{\nabla}\Psi \right\rangle = \sum_j Z_j e \Gamma_j$ is zero by virtue of charge conservation and Gauss' Law.[7] Thus any radial current stemming from NTV $\left[\left\langle \vec{J}_{\rm NTV} \cdot \vec{\nabla}\Psi \right\rangle \equiv \sum_j Z_j e \Gamma_j^{\rm na}\right]$, must be canceled out by the remaining radial flux-driven 'currents' in (2). In other words, no net current flows into (or out of) the bulk plasma as a result of NTV. This situation is in contrast to the case of ripple-trapping induced NTV[9] in the plasma edge, near the toroidal field coils. In the latter case, ambipolarity is violated $\left(\left\langle \vec{J} \cdot \vec{\nabla}\Psi \right\rangle \neq 0\right)$ and a direct loss of ions produces a measurable radial current *and* a $\vec{J}_r \times \vec{B}_\theta$ force on the plasma.

In general, $\Gamma_j^{\rm na}$ is driven by transit-time magnetic pumping [TTMP],[10,11] ripple trapping,[9] and radial drifts of banana orbits.[1] But when the effective collision frequency is less than the bounce frequency, $\nu_{\rm eff} < \omega_b$, drifting banana orbits usually dominate the other processes.

A strong correlation exists between the flow evolution physics of tokamaks and quasi-helically symmetric [QHS] stellarators such as HSX at the University of Wisconsin. In QHS-mode, $|B|$ varies to lowest order as $|B| = B_0(1 - \epsilon_h \cos M\alpha_h)$, with several much smaller helical 'sidebands.' The helical symmetry angle $\alpha_h \equiv \theta - N\zeta/M$, in which M, N are fixed integers, is analogous to the poloidal direction in tokamaks. Equivalent to the toroidal direction in a tokamak, there exists a direction of near helical symmetry and thus least flow damping along $\vec{e}_{\zeta_h}$ such that $\vec{e}_{\zeta_h} \cdot \vec{\nabla}\alpha_h = 0$. (Perfect helical 'axisymmetry' occurs when all sidebands are zero.) Because of this, flow damping and thus flow evolution in the two directions along a flux surface decouple[7].

Specifically, when $\delta B_{\rm eff}/B_0 \ll \epsilon$, where $\delta B_{\rm eff}/B_0$ is a measure of the effective variation in $|B|$ in the near symmetry direction and $\epsilon = r/R_0$ is a measure of the dominant variation of $|B|$ in a flux surface, the rate of change of rotation in the near symmetry direction owing to NTV is expressible as

$$\frac{\partial \Omega}{\partial t} = -\mu_\parallel \left(\frac{\delta B_{\rm eff}}{B_0}\right)^2 (\Omega - \Omega_*) . \quad (3)$$

Here Ω is the rotation rate along the symmetry direction, $\mu_\parallel$ is the NTV damping rate, and Ω_* is a diamagnetic-type 'offset' rotation, to which NTV damps the flow, analogous to the nonzero rotation rate caused by poloidal flow damping.[12] Recent experiments on the DIII-D tokamak have observed the rotation offset Ω_* for the first time[4]. While many experiments have confirmed[2–5] the general form of NTV damping given above (3), a detailed comparison between experiment and theory has yet to be made.

To better facilitate such theory-experiment comparisons, this report seeks to expand previously obtained asymptotic "low collisionality" NTV results[1] in three important ways. First, we present the NTV particle fluxes and torques in Clebsch coordinates (Ψ, θ, β), as opposed to Hamada coordinates employed previously.[1] The generalized coordinates are: a radial flux surface label (Ψ), a straight-field-line poloidal angle (θ), and a field-line label (β). Second, in Sec. V we construct a smoothed particle flux using a technique originally developed in axisymmetric neoclassical theory,[13] which transitions between the previous asymptotic results.[1] This will benefit present experiments since both regimes are observed in different regions of the plasma volume[4], and the more plausible parameter case intermediate between the two asymptotic limits was missing. Third, in Sec. VII we derive NTV for quasi-helically symmetric stellarator geometry to aide future experimental comparison of NTV in QHS-mode stellarators and tokamaks.

In Sec. II the bounce-averaged drift kinetic equation is presented using Clebsch coordinates. In Secs. III–IV the $1/\nu$- and ν-regimes are re-derived in the new coordinates. In Sec. VI we relate the new work in this report to the previously-obtained deeply asymptotic cases.[1] Finally, the main points of this report are summarized in Sec. VIII.

II. DRIFT KINETIC EQUATION IN CLEBSCH COORDINATES

We begin by considering the effect of small non-axisymmetric magnetic fields on a toroidally symmetric equilibrium. The non-axisymmetric fields are assumed to be sufficiently small that the magnetic topology remains toroidal (i.e., no magnetic islands are present). Calculating particle transport in such a geometry is greatly simplified by employing generalized Clebsch coordinates (Ψ, θ, β) where the total magnetic field (perturbed plus equilibrium) is written as

$$\vec{B} = \vec{\nabla}\Psi \times \vec{\nabla}\beta. \quad (4)$$

Here Ψ is the poloidal magnetic flux, with Ψ = constant defining a perturbed magnetic flux surface, and θ is a straight-field-line poloidal angle with periodicity 2π. In toroidal systems the Clebsch coordinate β is known as the field line label, and with our choice of radial variable being the poloidal flux, is expressible in straight-field-line coordinates (Ψ, θ, ζ) as

$$\beta = q(\Psi)\theta - \zeta. \quad (5)$$

Here ζ is the straight-field-line toroidal angle, and $q \equiv \vec{B}\cdot\vec{\nabla}\zeta/\vec{B}\cdot\vec{\nabla}\theta$ is the conventional tokamak safety factor. The covariant-base vectors in the Clebsch system are

$$\vec{e}_\Psi = J\,\vec{\nabla}\theta \times \vec{\nabla}\beta, \quad (6)$$

$$\vec{e}_\theta = J\,\vec{\nabla}\beta \times \vec{\nabla}\Psi, \quad (7)$$

$$\vec{e}_\beta = J\,\vec{\nabla}\Psi \times \vec{\nabla}\theta = -\vec{e}_\zeta, \quad (8)$$

where $J = 1/\left(\vec{\nabla}\Psi \cdot \vec{\nabla}\theta \times \vec{\nabla}\beta\right)$, and $\vec{B}\cdot\vec{\nabla}\theta = -1/J$. For simplicity, consider a model up/down symmetric tokamak field, whose magnetic field magnitude in the presence of field errors may be decomposed as

$$B = B_0(\theta) + B_0 \sum_{m,n} \left(b_{nmc}\cos\alpha_{m,n} + b_{nms}\sin\alpha_{m,n}\right), \quad (9)$$

$$B_0(\theta) = B_0\left(1 - \epsilon\cos\theta\right), \qquad \epsilon = r/R_0, \quad (10)$$

where the m, n helical angle is defined as $\alpha_{m,n} \equiv m\theta - n\zeta$, and the harmonic coefficients satisfy $b_{nmc}, b_{nms} \ll \epsilon$. In what follows, it is convenient to employ trigonometric identities to represent (9) in the more compact form

$$B = B_0(\theta) - B_0 \sum_n D_n(\theta, \beta), \quad (11)$$

where the harmonic amplitudes are given by

$$D_n(\theta,\beta) = A_n(\theta)\cos(n\beta) + B_n(\theta)\sin(n\beta), \quad (12)$$

$$A_n(\theta) = -\sum_m \left(b_{nmc}\cos\Delta_{m,n} + b_{nms}\sin\Delta_{m,n}\right), \quad (13)$$

$$B_n(\theta) = \sum_m \left(b_{nmc}\sin\Delta_{m,n} - b_{nms}\cos\Delta_{m,n}\right), \quad (14)$$

$$\Delta_{m,n} = (m - nq)\,\theta. \quad (15)$$

We are interested in calculating the non-ambipolar particle transport in the "low collisionality" limit $\nu_{\text{eff}} < \omega_b$, where $\nu_{\text{eff}} = \nu/\epsilon$ is the effective collision frequency for trapped particles, and $\omega_b = 2\pi/\tau_b$ is the bounce frequency. In this work, the bounce average and bounce time for trapped particles are defined respectively as:

$$\langle A\rangle_b = \frac{1}{\tau_b}\oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{B}{|v_\parallel|}A, \tag{16}$$

$$\tau_b = \oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{B}{|v_\parallel|}. \tag{17}$$

The integration $\oint d\theta = \int_{-\theta_b}^{\theta_b}$ is between symmetric bounce points. The bounce-averaged drift kinetic equation [DKE] governing $f_j(\Psi,\beta;\vec{v}_j)$ for species j is given by[14–16]

$$\dot{\Psi}\frac{\partial f_j}{\partial\Psi} + \dot{\beta}\frac{\partial f_j}{\partial\beta} = \langle C(f_j)\rangle_b\,, \tag{18}$$

$$\langle C(f_j)\rangle_b = \frac{\nu_D^j}{\tau_b}\frac{\partial}{\partial\mu}\left[\mu\,\frac{\partial f_j}{\partial\mu}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}m_j\,|v_\parallel|\right], \tag{19}$$

$$\nu_D^j = \sum_\beta \nu_{j\,,\beta}\left[\Phi(v/v_{t\,\beta}) - G(v/v_{t\,\beta})\right]\left(\frac{v_{t\,j}}{v}\right)^3. \tag{20}$$

Here the most probable thermal speed is defined by

$$v_{t\,\beta} = \sqrt{2T_\beta/m_\beta}, \tag{21}$$

the reference collision frequency[15] for species j is defined as $\nu_{j\,,\beta} = 4\pi n_\beta(Z_je)^2(Z_\beta e)^2\ln\Lambda_{j\,\beta}/\left([4\pi\epsilon_0]^2 m_j^2 v_{t\,j}^3\right)$, $G(x)$ is the Chandrasekhar function and $\Phi(x)$ is the error function. The magnetic moment appearing in (19) is $\mu \equiv m_j v_\perp^2/(2B)$. The generalized bounce-averaged drift frequencies (see Appendices A and B) are $\dot{\Psi} = \langle\vec{v}_d\cdot\nabla\Psi\rangle_b$ and $\dot{\beta} = \langle\vec{v}_d\cdot\nabla\beta\rangle_b$. For simplicity, we consider a quasineutral ion-electron plasma with comparable species temperatures, which reduces the perpendicular deflection frequency for species j (20) to:

$$\nu_D^j = \nu_{j\,,j}\,F(x;b_j)\,x^{-3/2}\,, \tag{22}$$

$$F(x;b_j) = \Phi(\sqrt{x}) - G(\sqrt{x}) + \Phi(\sqrt{b_jx}) - G(\sqrt{b_jx}), \tag{23}$$

where $x \equiv \mathcal{E}_j/T_j$ with $\mathcal{E}_j$ being the particle's kinetic energy, and $b_j = m_\beta/m_j$ (for example $b_i = m_e/m_i$, etc.).

We proceed in the standard fashion[16] by noting that the collision frequency is much larger than the "radial drift" frequency, and thus expand (18) in powers of $\delta \equiv \dot{\Psi}/[\nu_D^j\Delta\Psi]$ assuming for the moment that $\nu_D^j \sim \dot{\beta}/\Delta\beta$. The lowest order equation yields

$$\dot{\beta}\frac{\partial f_{j,(0)}}{\partial\beta} = \left\langle C(f_{j,(0)})\right\rangle_b\,, \tag{24}$$

which is satisfied by $f_0 = f_M(\Psi)$; i.e., a Maxwellian which is a function of the "radial" flux surface label Ψ only. The next order equation yields

$$\dot{\Psi}\frac{\partial f_{j,(0)}}{\partial\Psi} + \dot{\beta}\frac{\partial f_{j,(1)}}{\partial\beta} = \left\langle C(f_{j,(1)})\right\rangle_b\,, \tag{25}$$

which will be expanded further.

III. PARTICLE FLUX IN THE $1/\nu$ REGIME

Here we make an additional assertion that the "processional drift" is less than the collision frequency, and further expand $f_{j,(1)}$ in powers of $\delta_2 \equiv \dot{\beta}/[\nu_D^j\Delta\beta]$. When $\delta_2 \geq 1$ we will cross over into the ν regime to be discussed in the next section. Assuming, as is reasonable, that pitch-angle scattering dominates over energy scattering, we may discretize the energy component of phase space; whence the condition $\delta_2 = 1$ dictates an upper limit on particle energy for the $1/\nu$ regime. We approximate $\nu_D^j(x) \sim \nu_{j,j}\,x^{-3/2}$ (valid when $x = \mathcal{E}_j/T_j \gg 1$), and find the "critical" upper limit on energy to be given by

$$x_c \equiv \frac{\mathcal{E}_c}{T_j} = \left(\frac{\nu_{j,j}}{\epsilon\,\phi'(\Psi)}\right)^{2/3}, \tag{26}$$

where $\dot{\beta}$ is calculated from (B24) in the low plasma-beta limit. The lowest order in δ_2 equation,

$$\dot{\Psi}\frac{\partial f_{j,(0)}}{\partial\Psi} = \frac{\nu_D^j}{\tau_b}\frac{\partial}{\partial\mu}\left[\mu\,\frac{\partial f_{j,(1,0)}}{\partial\mu}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}m_j\,|v_\parallel|\right], \tag{27}$$

can be integrated directly using (B23) to yield

$$\frac{1}{Z_je}\frac{\partial f_{j,(0)}}{\partial\Psi}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{\partial B}{\partial\beta}|v_\parallel| = \nu_D^j\frac{\partial f_{j,(1,0)}}{\partial\mu}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}|v_\parallel|. \tag{28}$$

The flux-surface-averaged particle flux for species j is defined as

$$\Gamma \equiv \left\langle n_j\vec{u}_j\cdot\vec{\nabla}\Psi\right\rangle. \tag{29}$$

In the $1/\nu$ regime the particle flux is

$$\Gamma_{1/\nu} = \frac{1}{A}\int dS\int_{\mathcal{E}<\mathcal{E}_c} d^3v\;\vec{v}_d\cdot\vec{\nabla}\Psi\,f_j, \tag{30}$$

$$A = \int dS = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{d\theta d\beta}{\vec{B}\cdot\vec{\nabla}\theta}, \tag{31}$$

where the velocity space integration is carried out up to particles with energy $\mathcal{E}_c$. Following algebra laid out in Appendix C, the $1/\nu$ regime particle flux reduces to

$$\Gamma_{1/\nu\,j} = -\frac{\epsilon^{3/2}I_\lambda}{m_j^{3/2}(Z_je)^2}\int_0^{\mathcal{E}_c}\frac{f_0'(\Psi)\mathcal{E}^{5/2}}{\nu_D^j}d\mathcal{E}, \tag{32}$$

where $' = d/d\Psi$ and the integral over the perturbed magnetic field harmonics, I_λ, is defined by

$$I_\lambda \equiv \frac{16\pi^2}{A}\int_0^1 d\kappa^2[\hat{J}(\kappa)]^{-1}\sum_n n^2\left[a_n^2+b_n^2\right], \tag{33}$$

$$\hat{J}(\kappa) \equiv \oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\sqrt{\kappa^2-\sin^2\theta/2}. \tag{34}$$

The above integration is carried out over the normalized pitch angle parameter κ, defined by

$$\kappa^2 = \frac{1-\lambda+\lambda\epsilon}{2\lambda\epsilon}, \tag{35}$$

$$\lambda = \frac{\mu B_0}{\frac{1}{2}m_j v^2} \equiv \frac{\mu B_0}{\mathcal{E}}. \tag{36}$$

The bounce integral coefficients a_n and b_n over the magnetic field harmonics $A_n(\theta)$ and $B_n(\theta)$ respectively, are

$$a_n = \oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\sqrt{\kappa^2-\sin^2\theta/2}\; A_n(\theta), \tag{37}$$

$$b_n = \oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\sqrt{\kappa^2-\sin^2\theta/2}\; B_n(\theta). \tag{38}$$

Equation (24) dictates the equilibrium distribution function is a Maxwellian:

$$f_0 \equiv f_{j,(0)} = n(\Psi)\left(\frac{m_j}{2\pi T_j(\Psi)}\right)^{3/2} e^{-\mathcal{E}/T_j(\Psi)}, \tag{39}$$

dependent only on the flux coordinate Ψ. Differentiating with respect to Ψ yields

$$f_0' = f_0\left(\frac{p_j'}{p_j} + Z_j e\frac{\phi'}{T_j} + \frac{T_j'}{T_j}\left[\frac{\mathcal{E}}{T_j}-\frac{5}{2}\right]\right). \tag{40}$$

Thus, we see the drive for the non-ambipolar particle flux Γ_j is given by the usual radial thermodynamic forces p_j', T', and the radial electric field $-\phi'$. However, this can be recast in terms of flow frequencies by replacing the pressure gradient and electric field using radial force balance:

$$\frac{p_j'}{p_j} + Z_j e\frac{\phi'}{T_j} = Z_j e\frac{\vec{V}_j\cdot\vec{\nabla}\beta}{T_j}. \tag{41}$$

Using (5) to replace β in favor of the straight field line poloidal and toroidal angles this result simplifies (40) to

$$f_0' = -f_0\frac{Z_j e}{T_j}\left(\vec{V}_j\cdot\vec{\nabla}\zeta - \Omega_j(\mathcal{E})\right). \tag{42}$$

Here $\vec{V}_j\cdot\vec{\nabla}\zeta$ is a generalized "toroidal flow frequency" and

$$\Omega_j(\mathcal{E}) \equiv q\vec{V}_j\cdot\vec{\nabla}\theta + \frac{T_j'}{Z_j e}\left(\frac{\mathcal{E}}{T_j}-\frac{5}{2}\right), \tag{43}$$

is an energy dependent "shift" frequency. Thus, the particle flux can be thought of as driven either by thermodynamic forces and the radial electric field, or by the difference between the "toroidal flow frequency" $\vec{V}_j\cdot\vec{\nabla}\zeta$ and the shift frequency Ω_j. In terms of the thermodynamic forces, the particle flux in (32) is

$$\Gamma_{1/\nu j} = -C_{1/\nu j}\left[\left(\frac{p_j'}{p_j}+\frac{Z_j e\phi'}{T_j}\right)\lambda_{1j} + \frac{T_j'}{T_j}\lambda_{2j}\right], \tag{44}$$

$$C_{1/\nu j} = \frac{\epsilon^{3/2} I_\lambda}{\pi^{3/2}\sqrt{32}}\left(\frac{m_j}{Z_j e}\right)^2 \frac{n_j\, v_{tj}^4}{\nu_{j,j}}. \tag{45}$$

The kinetic integrals appearing above are

$$\lambda_{1j} = \frac{1}{2}\int_0^{\hat{\nu}_j^{2/3}} \frac{e^{-x}\, x^4\, dx}{F(x;b_j)}, \tag{46}$$

$$\lambda_{2j} = \frac{1}{2}\int_0^{\hat{\nu}_j^{2/3}} \frac{e^{-x}\, x^4(x-5/2)\, dx}{F(x;b_j)}. \tag{47}$$

Here $x_c = \hat{\nu}_j^{2/3}$ where the dimensionless transition frequency between the $1/\nu$ and ν regimes is

$$\hat{\nu}_j = \frac{\nu_{j,j}}{\epsilon\,\phi'(\Psi)}, \tag{48}$$

and $\phi'(\Psi)$ is the toroidal precession frequency [see (B24)]. The derivation of (44) can be found in appendix C.

In terms of the difference between the toroidal flow frequency $\vec{V}_j\cdot\vec{\nabla}\zeta$ and an energy-averaged shift frequency Ω_*, the particle flux is written

$$Z_j e\Gamma_{j1/\nu} = n_j m_j \mu_{\|j}^{1/\nu} I_\lambda\left(\left\langle R^2\vec{V}_j\cdot\vec{\nabla}\zeta\right\rangle - \left\langle R^2\Omega_{*j}^{1/\nu}\right\rangle\right), \tag{49}$$

$$\mu_{\|j}^{1/\nu} = \frac{\epsilon^{1/2}}{\sqrt{8\pi^3}}\frac{v_{tj}^2}{\langle R^2\rangle\,\omega_E}\frac{\lambda_{1j}(\hat{\nu}_j)}{\hat{\nu}_j}, \tag{50}$$

$$\left\langle R^2\Omega_{*j}^{1/\nu}\right\rangle = \left\langle R^2 q\vec{V}\cdot\vec{\nabla}\theta\right\rangle + \frac{c_{1/\nu}}{Z_j e}\frac{dT_j}{d\Psi}\left\langle R^2\right\rangle, \tag{51}$$

and $\langle\ldots\rangle$ denotes a flux surface average. The $1/\nu$ regime toroidal rotation coefficient is given by

$$c_{1/\nu} \equiv \frac{\lambda_{2j}(\hat{\nu}_j)}{\lambda_{1j}(\hat{\nu}_j)}. \tag{52}$$

As the particle energy increases, at some point equation (26) specifying the validity of the $1/\nu$ regime is violated, i.e., when $\delta_2 \geq 1$. When this occurs, the oscillatory nature of the radial drift combined with reduced collisions greatly reduce the radial transport.

IV. PARTICLE FLUX IN THE ν REGIME

For the case when $\delta_2 \geq 1$, we employ a subsidiary expansion for $f_{j,1}$ now using $1/\delta_2$ or explicity,

$$\delta_3 = \frac{\nu_D^j\Delta\beta}{\dot{\beta}} \ll 1, \tag{53}$$

as a small parameter.[1] The lowest order equation is then

$$\dot{\Psi}\frac{\partial f_{j,(0)}}{\partial\Psi} + \dot{\beta}\frac{\partial f_{j,(1,0)}}{\partial\beta} = 0, \tag{54}$$

which can be rewritten using (B23) and (B24) to yield

$$\frac{\partial f_{j,(1,0)}}{\partial\beta} \simeq \frac{f_0'(\Psi)m_j}{Z_j e\,\phi'(\Psi)\tau_b}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\left[-\frac{\partial B}{\partial\beta}\right]\frac{\partial}{\partial\mu}\left(\mu|v_\||\right). \tag{55}$$

Integrating this equation we arrive at the first order distribution function in the ν regime

$$f_{j,(1,0)} = \frac{f_0'(\Psi) m_j}{Z_j e\, \phi'(\Psi) \tau_b} \oint \frac{d\theta\, B}{\vec{B}\cdot\vec{\nabla}\theta} \times \left[|v_{\parallel 0}| - \frac{\mu B}{m_j |v_{\parallel 0}|} \right] \sum_n D_n(\theta,\beta). \quad (56)$$

The flux-surface-averaged particle flux for species j in the ν regime is

$$\Gamma_\nu = \frac{1}{A} \int dS \int_{\mathcal{E}_c<\mathcal{E}} d^3v\ \vec{v}_d \cdot \vec{\nabla}\Psi\, f_j. \quad (57)$$

Here the flux surface area A is defined in (31) and the velocity space integration is carried out for "tail" particles with energy greater than $\mathcal{E}_c$. Following algebra laid out in Appendix D, we find the particle flux in the ν regime expressed in terms of the thermodynamic forces to be

$$\Gamma_{\nu j} = -C_{\nu j} \left[\left(\frac{p_j'}{p_j} + \frac{Z_j e \phi'}{T_j} \right) \eta_{1j} + \frac{T_j'}{T_j} \eta_{2j} \right], \quad (58)$$

$$C_{\nu j} = \frac{G_\lambda}{\pi^{3/2}\sqrt{32}} \left(\frac{m_j}{Z_j e} \right)^2 \frac{n_j\, v_{tj}^4 \nu_{j,j}}{\epsilon^{1/2} (\phi')^2}, \quad (59)$$

where

$$G_\lambda \equiv \frac{16\pi^2}{A} \int_0^1 d\kappa^2 \hat{J}(\kappa) \sum_n \left[\alpha_n^2 + \beta_n^2 \right]. \quad (60)$$

The bounce integral coefficients α_n and β_n are defined as

$$\alpha_n = \frac{\partial}{\partial\kappa^2} \left(\frac{1}{2\hat{\tau}_b} \oint \frac{B d\theta}{\vec{B}\cdot\vec{\nabla}\theta} \frac{\left[3 \left| v_{\parallel,0}/v \right|^2 - 1 \right]}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} A_n(\theta) \right), \quad (61)$$

$$\beta_n = \frac{\partial}{\partial\kappa^2} \left(\frac{1}{2\hat{\tau}_b} \oint \frac{B d\theta}{\vec{B}\cdot\vec{\nabla}\theta} \frac{\left[3 \left| v_{\parallel,0}/v \right|^2 - 1 \right]}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} B_n(\theta) \right). \quad (62)$$

The κ derivatives appearing in these integrals are divergent as the trapped-passing boundary is approached: $\kappa \to 1$. By including a collisional boundary layer at $\kappa = 1$, the singular nature of the integrals is removed.[17] We will treat this aspect of the ν regime in a later paper employing a WKB-method to patch the different asymptotic regimes into a single smoothed particle flux function.

The normalized bounce time is given by

$$\hat{\tau}_b = \oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta} \frac{B}{\sqrt{\kappa^2 - \sin^2(\theta/2)}}, \quad (63)$$

and the kinetic integrals over the deflection frequency in

	λ_1	λ_2	η_1	η_2
ions	13.31	31.35	0.36	-0.09
electrons	6.39	15.59	0.85	-0.34

TABLE I: Asymptotic kinetic coefficients for deuterium/electron plasma with mass ratios $b_i = 1/3672$ and $b_e = 3672$.

the ν regime are

$$\eta_{1j} = \frac{1}{2} \int_{\hat{\nu}_j^{2/3}}^\infty e^{-x}\, x F(x; b_j) dx, \quad (64)$$

$$\eta_{2j} = \frac{1}{2} \int_{\hat{\nu}_j^{2/3}}^\infty e^{-x}\, x\,(x - 5/2)\, F(x; b_j) dx. \quad (65)$$

In terms of the difference between the toroidal flow frequency $\vec{V}_j \cdot \vec{\nabla}\zeta$ and an energy-averaged shift frequency Ω_*, the particle flux in the ν regime is

$$Z_j e \Gamma_{j\,\nu} = n_j m_j \mu_{\parallel j}^\nu G_\lambda \left(\left\langle R^2 \vec{V}_j \cdot \vec{\nabla}\zeta \right\rangle - \left\langle R^2 \Omega_{*j}^\nu \right\rangle \right), \quad (66)$$

$$\mu_{\parallel j}^\nu = \frac{\epsilon^{1/2}}{\sqrt{8\pi^3}} \frac{v_{tj}^2}{\langle R^2 \rangle \phi'(\Psi)} \eta_{1j}(\hat{\nu}_j)\, \hat{\nu}_j, \quad (67)$$

$$\left\langle R^2 \Omega_{*j}^\nu \right\rangle = \left\langle R^2 q \vec{V} \cdot \vec{\nabla}\theta \right\rangle + \frac{c_\nu}{Z_j e} \frac{dT_j}{d\Psi} \left\langle R^2 \right\rangle. \quad (68)$$

The ν regime toroidal rotation coefficient is given by

$$c_\nu \equiv \frac{\eta_{2j}(\hat{\nu}_j)}{\eta_{1j}(\hat{\nu}_j)}. \quad (69)$$

V. SMOOTHED PARTICLE FLUX

A smoothed non-ambipolar particle flux caused by non-axisymmetric magnetic field perturbations in the limit $\nu_{j,j}/\epsilon < \omega_b$ is constructed by adding the fluxes in both the $1/\nu$ (44) and ν (58) regimes:

$$\Gamma_j = -\frac{n_j m_j \left\langle R^2 \right\rangle}{(Z_j e)^2} \mu_{\parallel j} I_\lambda \left[\left(\frac{p_j'}{n_j} + Z_j e \phi' \right) + T_j' c_{tj} \right], \quad (70)$$

$$\mu_{\parallel j} = \frac{\epsilon^{1/2}}{\sqrt{8\pi^3}} \frac{v_{tj}^2}{\langle R^2 \rangle \phi'(\Psi)} \gamma_{1j}(\hat{\nu}_j), \qquad c_{tj} = \frac{\gamma_{2j}(\hat{\nu}_j)}{\gamma_{1j}(\hat{\nu}_j)}. \quad (71)$$

As will become clear later, $\mu_{\parallel j}$ is an effective NTV damping rate and c_{tj} a rotation coefficient, both with respect to the $\vec{e}_\zeta$ direction. Here $\vec{e}_\zeta \equiv J \vec{\nabla}\Psi \times \vec{\nabla}\theta = \vec{\nabla}\Psi \times \vec{\nabla}\theta / \vec{B}\cdot\vec{\nabla}\theta$, for the (Ψ, θ, ζ) coordinates. The smoothed kinetic coefficients are

$$\gamma_{1j} \equiv \frac{\lambda_{1j}}{\hat{\nu}_j} + \frac{G_\lambda}{I_\lambda} \eta_{1j} \hat{\nu}_j, \quad (72)$$

$$\gamma_{2j} \equiv \frac{\lambda_{2j}}{\hat{\nu}_j} + \frac{G_\lambda}{I_\lambda} \eta_{2j} \hat{\nu}_j. \quad (73)$$

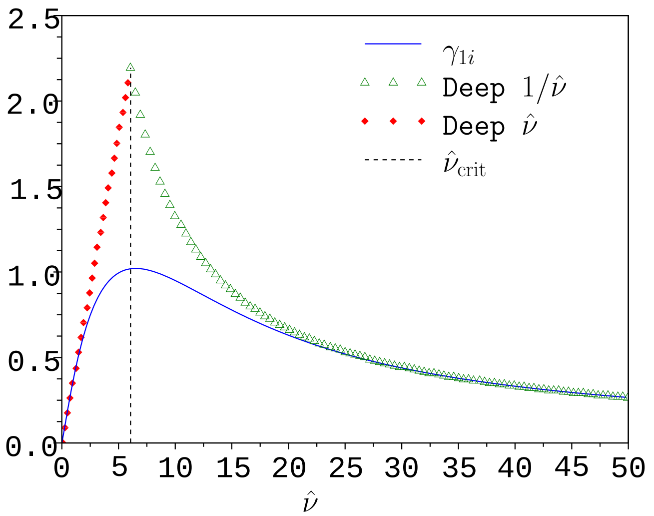

FIG. 1: Smoothed viscosity coefficient $\gamma_{1j}(\hat{\nu})$ given in (72), plotted for ions when $G_\lambda/I_\lambda = 1$.

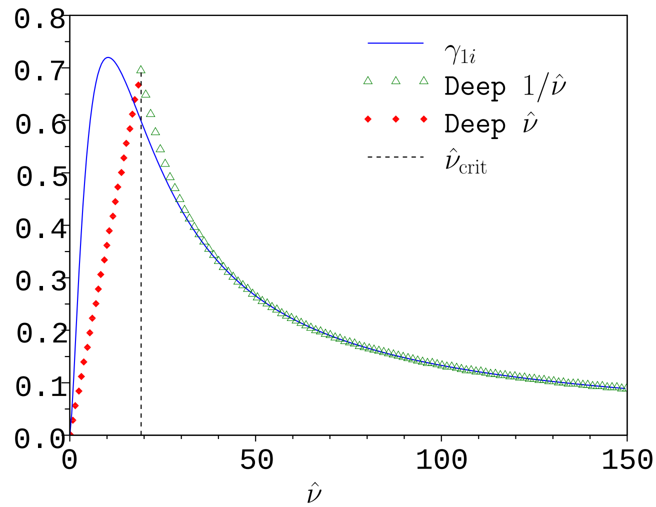

FIG. 2: Smoothed viscosity coefficient $\gamma_{1j}(\hat{\nu})$ given in (72), plotted for ions when $G_\lambda/I_\lambda = 0.1$.

The smoothed particle flux is related to the toroidal component of the parallel viscous stress tensor via

$$Z_j e \Gamma_j \equiv \left\langle n_j \vec{u}_j \cdot \vec{\nabla}\Psi \right\rangle = \left\langle \vec{e}_\zeta \cdot \vec{\nabla} \cdot \overleftrightarrow{\Pi}_j \right\rangle . \tag{74}$$

Hence, the toroidal flow evolution equation is given by

$$\left\langle n_j m_j \frac{\partial \vec{V}_j}{\partial t} \cdot \vec{e}_\zeta \right\rangle = - n_j m_j \mu_{\parallel j} I_\lambda \left(\left\langle R^2 \vec{V}_j \cdot \vec{\nabla}\zeta \right\rangle - \left\langle R^2 \Omega_{*j} \right\rangle \right), \tag{75}$$

where the smoothed toroidal rotation offset rate is

$$\left\langle R^2 \Omega_{*j} \right\rangle = \left\langle R^2 q \vec{V} \cdot \vec{\nabla}\theta \right\rangle + \frac{c_{tj}}{Z_j e} \frac{dT_j}{d\Psi} \left\langle R^2 \right\rangle . \tag{76}$$

VI. COMPARISON WITH PREVIOUS ASYMPTOTIC REGIMES

To recover the previous asymptotic results found in reference [1] we employ standard Hamada coordinates (V, θ, ζ) with unit Jacobian. The analysis in the preceding sections holds, with the identification that the Clebsch coordinate $\Psi = \chi$, the poloidal magnetic flux function. The poloidal and toroidal magnetic flux functions are defined such that $\vec{B} \cdot \vec{\nabla}\theta = \chi'(V)$ and $\vec{B} \cdot \vec{\nabla}\zeta = \psi'(V)$ respectively. In reference [1] the particle flux is defined by

$$\Gamma^S = \left\langle n_j \vec{u}_j \cdot \vec{\nabla} V \right\rangle , \tag{77}$$

and thus we must divide the flux defined in this work by χ'. After some algebra, (70) becomes

$$\Gamma_j^S = \frac{\Gamma_j}{\chi'} = - \frac{n_j m_j^2}{(Z_j e \chi')^2} \frac{\epsilon^{1/2}}{\sqrt{32\pi^3}} \frac{v_{tj}^4}{(\phi'/\chi')} \hat{I}_\lambda \times \left[\left(\frac{p_j'}{p_j} + \frac{Z_j e}{T_j}\phi' \right) \gamma_{1j}(\hat{\nu}_j) + \frac{T_j'}{T_j} \gamma_{2j}(\hat{\nu}_j) \right], \tag{78}$$

where

$$\hat{I}_\lambda \equiv \int_0^1 d\kappa^2 [E(\kappa) - (1-\kappa^2)K(\kappa)]^{-1} \times \sum_n n^2 \left[\hat{a}_n^2 + \hat{b}_n^2 \right] . \tag{79}$$

Here $' = \partial/\partial V$, $\hat{a}_n = \chi' a_n$ with a_n defined in (61), and likewise for $\hat{b}_n$. In the limit $\hat{\nu}_j \gg 1$, the plasma is deeply in the $1/\nu$ regime and the smoothed formula above reduces to

$$\Gamma_{j1/\nu}^S = - \frac{n_j m_j^2}{(Z_j e \chi')^2} \frac{\epsilon^{3/2}}{\sqrt{32\pi^3}} \frac{v_{tj}^4}{\nu_{j,j}} \hat{I}_\lambda \times \left[\left(\frac{p_j'}{p_j} + \frac{Z_j e}{T_j}\phi' \right) \lambda_{1j} + \frac{T_j'}{T_j} \lambda_{2j} \right], \tag{80}$$

exactly equation (16) in[1] albeit in SI units instead of cgs. The asymptotic kinetic integrals are defined by $\lambda_{1j}(\hat{\nu}_j) \to \lambda_{1j}$ in the limit $\hat{\nu}_j \to \infty$ and likewise for λ_{2j}. For a two species deuterium/electron plasma the asymptotic kinetic integral values are given in table I. The results for electrons are in agreement with reference [1], however the ion numbers disagree slightly because the electron contribution to the ion deflection frequency was neglected in reference [1].

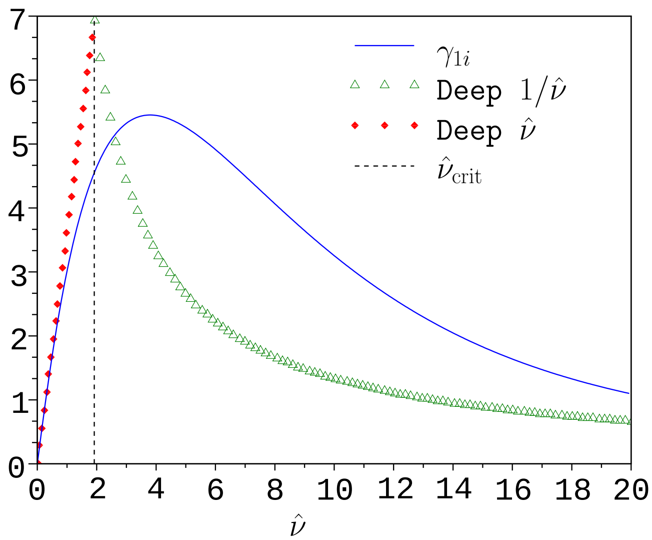

FIG. 3: Smoothed viscosity coefficient $\gamma_{1j}(\hat{\nu})$ given in (72), plotted for ions when $G_\lambda/I_\lambda = 10$.

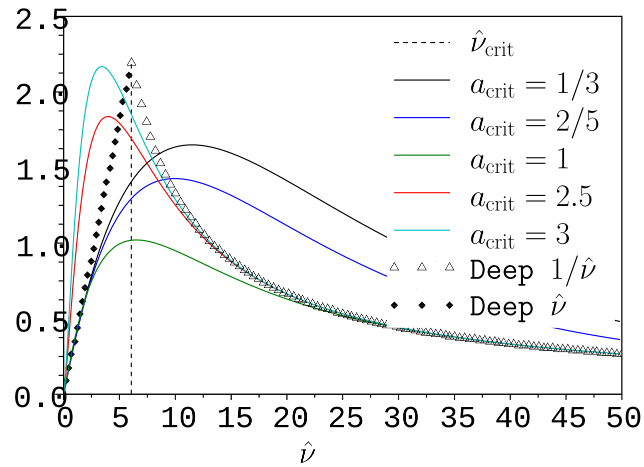

FIG. 4: Demonstration of extreme sensitivity in the integral truncation method. Plotted are values of $\gamma_{1j}(\hat{\nu})$ given in (72), for ions when $G_\lambda/I_\lambda = 1$, and when the truncation between regimes is done for $x = (a_{crit}\hat{\nu})^{2/3}$, not $x = \hat{\nu}^{2/3}$.

In the opposite limit $\hat{\nu}_j \ll 1$, the plasma is deeply in the ν regime and (78) reduces to

$$\Gamma^S_{j\nu} = -\frac{n_j m_j^2}{(Z_j e\chi')^2}\frac{\epsilon^{-1/2}}{\sqrt{32\pi^3}}\frac{v_{tj}^4 \nu_{j,j}}{\omega_E^2}\hat{G}_\lambda \times \left[\left(\frac{p_j'}{p_j} + \frac{Z_j e}{T_j}\phi'\right)\eta_{1j} + \frac{T_j'}{T_j}\eta_{2j}\right], \quad (81)$$

where here the toroidal precessional drift is defined as $\omega_E \equiv \phi'/\chi'$. The dimensionless pitch angle integral in the ν regime is defined

$$\hat{G}_\lambda = \int_0^1 d\kappa^2 [E(\kappa) - (1-\kappa^2)K(\kappa)] \sum_n \left[\hat{\alpha}_n^2 + \hat{\beta}_n^2\right], \quad (82)$$

$$\hat{\alpha}_n = \frac{\partial}{\partial\kappa^2}\left(\frac{1}{2K(\kappa)}\oint d\theta \frac{\left[3\left|v_{\parallel,0}/v\right|^2 - 1\right]}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} A_n(\theta)\right), \quad (83)$$

$$\hat{\beta}_n = \frac{\partial}{\partial\kappa^2}\left(\frac{1}{2K(\kappa)}\oint d\theta \frac{\left[3\left|v_{\parallel,0}/v\right|^2 - 1\right]}{\sqrt{\kappa^2 - \sin^2(\theta/2)}} B_n(\theta)\right). \quad (84)$$

Equation (81) is exactly (24) in reference [1], while (82)–(84) reduce to (25) in reference [1], with the exception of a typesetting error. [$G_\lambda \propto (\epsilon\, 32)^{-1/2}$, not $(\epsilon/32)^{-1/2}$ as is written in equation (25) in reference [1]. This typo is corrected in citation 2 of reference [17].] The asymptotic kinetic integrals are defined $\eta_{1j}(\hat{\nu}_j) \to \eta_{1j}$ in the limit $\hat{\nu}_j \to 0$ and likewise for η_{2j} with the numerical results given in table I.

Unfortunately, this method of integral truncation to patch different asymptotic regimes together is sensitive to where the truncation is performed.[18] Notice that in Figures 1-3 the peak in the particle flux moves around relative to the piecewise patch of the asymptotic limits depending on the ratio G_λ/I_λ. While it might appear that this is a physics result, we can make the peak move around more by adjusting where the truncation is made. For example, instead of truncating the $1/\nu$ energy integrals (46)–(47) and starting the ν energy integrals (64)–(65) at $\hat{\nu}^{2/3}$ we instead make the transition at $(a_{crit}\hat{\nu})^{2/3}$, where a_{crit} is a constant. The results are plotted in Figure 4 with $G_\lambda/I_\lambda = 1$. We find that a modest change in the truncation moves the peak around rather substantially. Since the previous theory[1] is asymptotic in nature, a precise determination of the truncation point in the energy integrals is not possible and a better method of patching the integrals together is warranted. To this end we will present a WKB-method in a later report which not only captures the asymptotic regimes here, but also reproduces the collisional boundary-layer at the trapped-passing boundary in the ν regime.[17]

VII. QUASI-HELICALLY SYMMETRIC STELLARATORS

Although the previous analysis is done in generalized Clebsch coordinates, the equilibrium magnetic field (9) is that of a tokamak. However, a strong correlation exists between tokamaks and quasi-helically symmetric [QHS] stellarators such as HSX[19] at the University of Wisconsin. The analogous Fourier-decomposed equation for $|B|$, i.e. Eqn. (9), in a QHS stellarator is

$$B = B_0\left(1-\epsilon_h \cos\alpha_{M,N}\right) + B_0 \sum_{\substack{m=0\\ m,n\neq M,N}}^{\infty} \sum_{n=-\infty}^{\infty} b_{nmc}\cos\alpha_{m,n}\ . \tag{85}$$

Notice this equilibrium possesses a dominant helical symmetry angle $\alpha_{M,N} = M\theta - N\zeta$, with poloidal and toroidal mode numbers M, N and several much smaller helical sidebands: $b_{nmc} \ll \epsilon_h$. For example, in HSX the poloidal and toroidal mode numbers are $M = 1$, and $N = 4$.[19] (Note: stellarator symmetry ensures that all $b_{nms} = 0$.) A compact mapping from tokamak to QHS mode can be made by the following straight-field-line coordinates $(\psi_h, \alpha_h, \zeta_h)$:

$$\psi_h = \int [M - Nq(\Psi)]\, d\Psi, \tag{86}$$

$$\alpha_h = \theta - \frac{N}{M}\zeta, \tag{87}$$

$$\zeta_h = \frac{\zeta}{M}, \tag{88}$$

where $q = \vec{B}\cdot\vec{\nabla}\zeta/\vec{B}\cdot\vec{\nabla}\theta$ is the usual toroidal safety factor. Furthermore, the helical safety factor and field line label are given by:

$$q_h \equiv \frac{\vec{B}\cdot\vec{\nabla}\zeta_h}{\vec{B}\cdot\vec{\nabla}\alpha_h} = \frac{q}{M-Nq}, \tag{89}$$

$$\beta_h \equiv q_h\alpha_h - \zeta_h = \frac{\beta}{M-Nq}. \tag{90}$$

In these coordinates the vector representation of the magnetic field is

$$\vec{B} = \vec{\nabla}\psi_h \times \vec{\nabla}\beta_h, \tag{91}$$

which using (86) and (90) reduces trivially to (4). The covariant-base vectors in this system are $\vec{e}_{\psi_h} = J\,\vec{\nabla}\alpha_h \times \vec{\nabla}\zeta_h$, $\vec{e}_{\alpha_h} = J\,\vec{\nabla}\zeta_h \times \vec{\nabla}\psi_h$, and $\vec{e}_{\zeta_h} = J\,\vec{\nabla}\psi_h \times \vec{\nabla}\alpha_h$. Finally, the Jacobian determinant is

$$J \equiv \frac{1}{\vec{\nabla}\psi_h\cdot\vec{\nabla}\alpha_h\times\vec{\nabla}\zeta_h} = \frac{1}{\vec{B}\cdot\vec{\nabla}\alpha_h}. \tag{92}$$

The Fourier sum of $|B|$ in Eqn. (85) can be expressed in terms of the helical coordinates (86)–(88) as

$$B = B_0\left[1-\epsilon_h\cos\left(M\alpha_h\right)\right] + B_0 \sum_{\substack{m=0\\ m,n'\neq M,0}}^{\infty} \sum_{n'=-\infty}^{\infty} b_{n'mc}\cos\left(m\alpha_h - n'\zeta_h\right), \tag{93}$$

where $n' \equiv nM - Nm$ and the prime will be dropped from now on. The flux-surface integration transforms from standard toroidal straight-line-coordinates to the helical coordinates defined here via the following mapping

$$\int_{-\pi}^{\pi} d\zeta \int_{-\pi}^{\pi} \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta} = \int_{-\pi/M}^{\pi/M} d\zeta_h \int_{-\pi-N\zeta_h}^{\pi-N\zeta_h} \frac{d\alpha_h}{\vec{B}\cdot\vec{\nabla}\alpha_h}, \tag{94}$$

$$= \frac{1}{M}\int_{-\pi}^{\pi} d\zeta_h \int_{-\pi/M}^{(2M-1)\pi/M} \frac{d\alpha_h}{\vec{B}\cdot\vec{\nabla}\alpha_h}, \tag{95}$$

where the last step is made using the doubly-periodic geometry of a torus. It is clear then, that we may define the flux-surface-average in helical coordinates as

$$\langle \dots \rangle_h \equiv \frac{1}{A_h}\int dS_h\left(\dots\right), \tag{96}$$

$$A_h \equiv \int_{-\pi}^{\pi} d\zeta_h \int_{-\pi/M}^{(2M-1)\pi/M} \frac{d\alpha_h}{\vec{B}\cdot\vec{\nabla}\alpha_h}, \tag{97}$$

where the limits of integration in (96) are the same as those in (97). The magnetic field strength given in (93) possesses M magnetic mirrors along the α_h coordinate. This must be taken into account when interchanging the order of integration between magnetic moment coordinate, μ, and symmetry angle, α_h in the calculation of the particle flux. Analogous to the weak poloidal mirror case of a tokamak [see Eqn. (C2)], we find

$$\int_{-\pi/M}^{(2M-1)\pi/M} d\alpha_h \int_{\mathcal{E}/B_M}^{\mathcal{E}/B(\alpha_h)} d\mu = \int_{\mathcal{E}/B_M}^{\mathcal{E}/B_m} d\mu \sum_{\alpha_c} \oint d\alpha_h, \tag{98}$$

where the sum over turning points is:

$$\sum_{\alpha_c} \oint d\alpha_h \equiv \sum_{j=0}^{M-1} \int_{\alpha_j^-}^{\alpha_j^+} d\alpha_h, \tag{99}$$

$$\alpha_j^{\pm} \equiv (2j)\,\frac{\pi}{M} \pm \alpha_c, \tag{100}$$

$$\alpha_c \equiv \frac{2}{M}\sin^{-1}\left(\kappa_h\right) \text{ with } 0 \le \alpha_c \le \pi/M. \tag{101}$$

Here κ_h is the pitch angle parameter defined in (35) with ϵ replaced by ϵ_h, and the principal turning point α_c is determined by $|v_{\|,0}|_h = 0$ [see Eqn. (127)]. Thus in quasi-helical symmetry with M magnetic mirrors, the bounce average defined in (16) generalizes to

$$\langle A\rangle_b \equiv \frac{1}{M\tau_b}\sum_{\alpha_c}\oint \frac{d\alpha_h}{\vec{B}\cdot\vec{\nabla}\alpha_h}\frac{B}{|v_\||}A, \tag{102}$$

where the bounce time over a single magnetic mirror is

$$\tau_b \equiv \oint \frac{d\alpha_h}{\vec{B}\cdot\vec{\nabla}\alpha_h}\frac{B}{|v_\||}, \tag{103}$$

and $\oint d\alpha_h \equiv \int_{-\alpha_c}^{\alpha_c} d\alpha_h$.

As in the tokamak case, it is convenient to employ trigonometric identities to represent (93) in the more compact form

$$B = B_0(\alpha_h) - B_0 \sum_n D_{nh}(\alpha_h, \beta_h), \tag{104}$$

where the Fourier harmonics are

$$D_{nh}(\alpha_h, \beta_h) \equiv A_{nh} \cos(n\beta_h) + B_{nh} \sin(n\beta_h), \tag{105}$$

$$A_{nh} \equiv -\sum_m b_{nmc} \cos \mathcal{D}_{mn}, \tag{106}$$

$$B_{nh} \equiv -\sum_m b_{nmc} \sin \mathcal{D}_{mn}, \tag{107}$$

$$\mathcal{D}_{mn} \equiv (m - nq_h)\alpha_h. \tag{108}$$

Now we are in a position to map the smoothed particle flux from a tokamak equilibrium to a quasi-helically-symmetric one. Analogous to the tokamak definition of the non-ambipolar particle flux (29) the helical particle flux is

$$\Gamma_h \equiv \left\langle n_j \vec{u}_j \cdot \vec{\nabla}\psi_h \right\rangle = (M - Nq)\,\Gamma_{\text{tok}}. \tag{109}$$

In terms of a viscous damping force, the smoothed helical particle flux becomes

$$Z_j e \Gamma_{hj} = \left\langle \vec{e}_{\zeta_h} \cdot \vec{\nabla} \cdot \overleftrightarrow{\Pi}_j \right\rangle \tag{110}$$

$$= n_j m_j \mu_{\parallel hj} I_{\lambda h} \left(\left\langle \vec{V}_j \cdot \vec{\nabla}\zeta_h R^2 \right\rangle - \left\langle \Omega_{*hj} R^2 \right\rangle \right), \tag{111}$$

$$\mu_{\parallel hj} = \frac{\epsilon_h^{1/2}}{\sqrt{8\pi^3}} \frac{v_{tj}^2}{\langle R^2 \rangle \phi'(\psi_h)} \gamma_{1j}(\hat{\nu}_{jh}), \tag{112}$$

$$\left\langle \Omega_{*hj} R^2 \right\rangle = q_h \left\langle R^2 \vec{V} \cdot \vec{\nabla}\alpha_h \right\rangle + \frac{c_{hj}}{Z_j e} \frac{dT_j}{d\psi_h} \left\langle R^2 \right\rangle,$$

$$\equiv \frac{c_{\alpha j} + c_{hj}}{Z_j e} \frac{dT_j}{d\psi_h} \left\langle R^2 \right\rangle, \tag{113}$$

$$c_{hj} = \frac{\gamma_{2j}(\hat{\nu}_{jh})}{\gamma_{1j}(\hat{\nu}_{jh})}. \tag{114}$$

Notice the damping direction is

$$\vec{e}_{\zeta_h} = M\vec{e}_\zeta + N\vec{e}_\theta, \tag{115}$$

where $\vec{e}_\zeta, \vec{e}_\theta$ are the toroidal and poloidal covariant-base vectors given in (8) and (6). By construction the damping direction from non-quasi-symmetric magnetic fields is perpendicular to the direction of helical symmetry:

$$\vec{e}_{\zeta_h} \cdot \vec{\nabla}(M\alpha_h) = 0, \tag{116}$$

as in a weakly non-axisymmetric tokamak. The smoothed kinetic coefficients in quasi-helical symmetry are

$$\gamma_{1j} \equiv \frac{\lambda_{1j}}{\hat{\nu}_{jh}} + \frac{G_{\lambda h}}{I_{\lambda h}} \eta_{1j} \hat{\nu}_{jh}, \tag{117}$$

$$\gamma_{2j} \equiv \frac{\lambda_{2j}}{\hat{\nu}_{jh}} + \frac{G_{\lambda h}}{I_{\lambda h}} \eta_{2j} \hat{\nu}_{jh}. \tag{118}$$

The $\lambda_{1,2j}$ and $\eta_{1,2j}$ kinetic integrals above are given by (46)–(47) and (64)–(65) respectively, with the transition parameter $\hat{\nu}_j$ replaced by

$$\hat{\nu}_{jh} = \frac{\nu_{j,j}}{\epsilon_h \, \phi'(\psi_h)}. \tag{119}$$

Analogous to the tokamak case, the pitch angle integral and bounce coefficients stemming from the $1/\nu$-regime are

$$I_{\lambda h} \equiv \frac{16\pi^2}{A_h} \int_0^1 d\kappa_h^2 [\hat{J}_h(\kappa_h)]^{-1} \sum_n n^2 \left[a_n^2 + b_n^2\right], \tag{120}$$

$$\hat{J}_h(\kappa_h) \equiv \sum_{\alpha_c} \oint \frac{d\alpha_h}{\vec{B} \cdot \vec{\nabla}\alpha_h} \sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}, \tag{121}$$

$$a_{nh} \equiv \sum_{\alpha_c} \oint \frac{d\alpha_h}{\vec{B} \cdot \vec{\nabla}\alpha_h} \sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}\; A_{nh}(\alpha_h), \tag{122}$$

$$b_{nh} \equiv \sum_{\alpha_c} \oint \frac{d\alpha_h}{\vec{B} \cdot \vec{\nabla}\alpha_h} \sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}\; B_{nh}(\alpha_h). \tag{123}$$

Finally, the pitch angle integral and bounce coefficients stemming from the ν-regime are

$$G_{\lambda h} \equiv \frac{16\pi^2}{A_h} \int_0^1 d\kappa_h^2 \hat{J}(\kappa_h) \sum_n \left[\alpha_{nh}^2 + \beta_{nh}^2\right], \tag{124}$$

$$\alpha_{nh} \equiv \frac{\partial}{\partial \kappa_h^2} \left(\frac{1}{2M\hat{\tau}_b} \sum_{\alpha_c} \oint \frac{B d\alpha_h}{\vec{B} \cdot \vec{\nabla}\alpha_h} \times \frac{\left[3 \left|v_{\parallel,0}/v\right|_h^2 - 1\right]}{\sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}} A_{nh}(\alpha_h) \right), \tag{125}$$

$$\beta_{nh} \equiv \frac{\partial}{\partial \kappa_h^2} \left(\frac{1}{2M\hat{\tau}_b} \sum_{\alpha_c} \oint \frac{B d\alpha_h}{\vec{B} \cdot \vec{\nabla}\alpha_h} \times \frac{\left[3 \left|v_{\parallel,0}/v\right|_h^2 - 1\right]}{\sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}} B_{nh}(\alpha_h) \right). \tag{126}$$

As in the tokamak case, at the end of our calculation we approximate the particle orbits by their equilibrium value. Thus in helical coordinates $|v_{\parallel,0}|$ and the normalized bounce time for one magnetic well are given by

$$\frac{|v_{\parallel,0}|_h}{v} \equiv \sqrt{2\lambda\epsilon_h} \sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}, \tag{127}$$

$$\hat{\tau}_b \equiv \oint \frac{d\alpha_h}{\vec{B} \cdot \vec{\nabla}\alpha_h} \frac{B}{\sqrt{\kappa_h^2 - \sin^2(M\alpha_h/2)}}. \tag{128}$$

Similar to the tokamak derivation, the κ_h derivatives appearing in the ν-regime pitch angle integrals are divergent as the trapped-passing boundary is approached:[17]

$\kappa_h \to 1$. This aspect of the ν regime will be treated in a later paper employing a WKB-method to patch the different asymptotic regimes into a single smoothed particle flux function.

VIII. SUMMARY

In this report, radial particle fluxes induced by non-axisymmetric magnetic perturbations for the low-collisionality "ν" and "$1/\nu$" banana-drift regimes previously calculated by K.C. Shaing[1] are unified into a single particle flux (or toroidal viscous force). Provided pitch-angle scattering dominates over collisional energy exchange, the energy component of phase space can be decoupled into independent regions [$E > E_c$ for ν regime, $E < E_c$ for $1/\nu$ regime, with E_c determined by $\nu_i(E_c) = \epsilon\,\omega_E$] [see Eqn. (26)] within which the perturbed distribution function can be calculated as in Shaing's work. Using a technique first employed in axisymmetric neoclassical theory,[13] the smoothed particle flux is constructed by summing the partial contributions from ν and $1/\nu$ banana drift effects respectively. The result is expressed in terms of thermodynamic particle forces in Eqn. (70), or using radial force balance [see Eqn. (41)] in terms of flux-surface flows in Eqn. (75).

In Sec. VII we have extended the smoothed NTV calculation from a tokamak to a quasi-helical-symmetric stellarator equilibrium with symmetry angle $M\theta - N\zeta$. This geometry possesses an analogous "NTV-like" torque in the $\vec{e}_{\zeta_h} = M\vec{e}_\zeta + N\vec{e}_\theta$ direction stemming from particles undergoing radial banana-drift transport in M magnetic wells along the symmetry coordinate $\alpha_h = \theta - N\zeta/M$, similar to the poloidal coordinate in tokamaks.

We find the method of integral truncation to patch together different asymptotic collisionality regimes is sensitive to where the truncation is performed. As a demonstration, we truncated the energy integrals (46)–(47) and (64)–(65) at $(a_{\rm crit}\hat{\nu})^{2/3}$ instead of $\hat{\nu}^{2/3}$. The results are plotted in Figure 4 and clearly demonstrate the peak NTV torque can be moved around rather easily with a modest choice of the constant a_c. Since the previous theory[1] is asymptotic in nature, a precise determination of the truncation point in the energy integrals is not possible and a better method of patching the integrals together is warranted. To this end we will present a WKB-method in a later report which not only captures the asymptotic regimes here, but also reproduces the collisional boundary-layer at the trapped-passing boundary in the ν regime.[17]

Acknowledgments

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Appendix A: Guiding center drift velocity

Neglecting the inductive electric field, the guiding center drift velocity is

$$\vec{v}_{d\,j} \simeq \frac{1}{Z_j e B^2}\vec{B}\times\left[Z_j e\vec{\nabla}\phi + \mu\vec{\nabla}B + m v_\parallel^2\left(\vec{b}\cdot\vec{\nabla}\vec{b}\right)\right], \tag{A1}$$

where ϕ is the electrostatic potential. Using $\vec{b}\cdot\vec{\nabla}\vec{b} = -\vec{b}\times\vec{\nabla}\times\vec{b}$, together with MHD force balance gives $\vec{b}\cdot\vec{\nabla}\vec{b} = (1/B)\vec{\nabla}_\perp B + \mu_0\vec{\nabla}P/B^2 \sim (1/B)\vec{\nabla}_\perp B + \mathcal{O}(\beta)$. Neglecting the order β term yields

$$\vec{v}_{d\,j} = \frac{1}{Z_j e B^2}\vec{B}\times\left(Z_j e\vec{\nabla}\phi + \mu\vec{\nabla}B + \frac{m v_\parallel^2}{B}\vec{\nabla}B\right). \tag{A2}$$

From energy conservation, we have that $0 = m v_\parallel\vec{\nabla}v_\parallel + \mu\vec{\nabla}B + Z_j e\vec{\nabla}\phi$; thus the drift velocity reduces to the well known result

$$\vec{v}_{d\,j} = \frac{v_\parallel}{B}\vec{\nabla}\rho_\parallel\times\vec{B}. \tag{A3}$$

Here, $\rho_\parallel = v_\parallel/(Z_j e B/m_j)$ is the gyro-radius calculated with the parallel component of velocity.

Appendix B: Bounce-averaged drift velocity

1. Heuristic derivation

In this appendix we provide a heuristic derivation of the well-known low-beta bounce-averaged particle drift velocity which is employed throughout this paper. It is advantageous to use a general Clebsch representation for the magnetic field with the third coordinate being the length l along a field line. Let the right-handed Clebsch coordinates be given by (Ψ, β, l) where Ψ and β define the magnetic field $\vec{B} = \vec{\nabla}\Psi\times\vec{\nabla}\beta$ in the usual way. The contravariant-base vectors are given by $\vec{e}^{\,\Psi} = \vec{\nabla}\Psi$, $\vec{e}^{\,\beta} = \vec{\nabla}\beta$, $\vec{e}^{\,l} = \vec{\nabla}l$, while the covariant-base vectors are $\vec{e}_\Psi = J\,\vec{\nabla}\beta\times\vec{\nabla}l$, $\vec{e}_\beta = J\,\vec{\nabla}l\times\vec{\nabla}\Psi$, and $\vec{e}_l = J\,\vec{\nabla}\Psi\times\vec{\nabla}\beta$. Here $J = 1/\left(\vec{\nabla}\Psi\cdot\vec{\nabla}\beta\times\vec{\nabla}l\right)$ is the Jacobian. By construction, our base vectors satisfy $\vec{e}^{\,i}\cdot\vec{e}_j = \delta^i_j$. The contravariant form for the magnetic field is $\vec{B} = \vec{e}_l/J = B^l\vec{e}_l$, and since l measures the distance along a field line $\vec{B}\cdot\vec{\nabla}l = B\hat{b}\cdot\vec{\nabla}l = B$. This immediately implies $J = 1/B$, $\hat{b} = \vec{e}_l$, and thus $B_l = \vec{B}\cdot\vec{e}_l = B$ as well. In this representation, we define the bounce average for trapped particles as

$$\langle A\rangle_b = \frac{1}{\tau_b}\oint\frac{dl}{|v_\parallel|}A, \tag{B1}$$

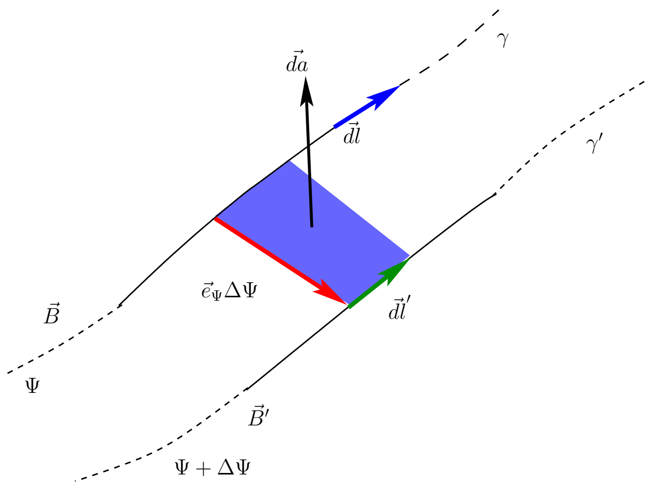

FIG. 5: Geometry for calculating the gradient of a path integral defined in Sec. B 2.

where $\tau_b = \oint dl/|v_\parallel|$, and take the integration to be over symmetric bounce points for simplicity: $\oint dl = \int_{-l_b}^{+l_b} dl$. We begin by bounce averaging the "radial" component of the drift velocity:

$$\left\langle \vec{v}_{d\,j} \cdot \vec{\nabla}\Psi \right\rangle_b = \frac{m_j}{Z_j e\,\tau_b} \oint dl \frac{1}{B} \vec{\nabla} \left(\frac{|v_\parallel|}{B} \right) \cdot \vec{B} \times \vec{\nabla}\Psi. \quad \text{(B2)}$$

Using the covariant form of $\vec{B}$ to express $\vec{B} \times \vec{\nabla}\Psi = B^2 \vec{e}_\beta - B B_\beta \vec{e}_l$, the above integral reduces to

$$\left\langle \vec{v}_{d\,j} \cdot \vec{\nabla}\Psi \right\rangle_b = \frac{m_j}{Z_j e\,\tau_b} \oint dl \left[B \frac{\partial}{\partial \beta} \left(\frac{|v_\parallel|}{B} \right) + \frac{\partial B_\beta}{\partial l} \frac{|v_\parallel|}{B} \right]. \quad \text{(B3)}$$

In low-β equilibria $\vec{J}$ is parallel to $\vec{B}$; thus both $\vec{\nabla} \times \vec{B} \cdot \vec{\nabla}\Psi$ and $\vec{\nabla} \times \vec{B} \cdot \vec{\nabla}\beta$ are zero. Expanding $\vec{\nabla} \times \vec{B} \cdot \vec{\nabla}\Psi = 0$ with the covariant form for the magnetic field yields $\partial B_l / \partial \beta = \partial B_\beta / \partial l$, and we find (B3) reduces to

$$\left\langle \vec{v}_{d\,j} \cdot \vec{\nabla}\Psi \right\rangle_b \equiv \dot{\Psi} = \frac{m_j}{Z_j e\,\tau_b} \frac{\partial}{\partial \beta} \oint dl\,|v_\parallel|. \quad \text{(B4)}$$

Similar logic and expanding $\vec{\nabla} \times \vec{B} \cdot \vec{\nabla}\beta = 0$ yields

$$\left\langle \vec{v}_{d\,j} \cdot \vec{\nabla}\beta \right\rangle_b \equiv \dot{\beta} = -\frac{m_j}{Z_j e\,\tau_b} \frac{\partial}{\partial \Psi} \oint dl\,|v_\parallel|. \quad \text{(B5)}$$

In vector form, the bounce-averaged low-beta particle drift velocity is compactly written as

$$\langle \vec{v}_{d\,j} \rangle_b = \frac{\vec{\nabla} J \times \vec{B}}{Z_j e\,\tau_b B^2}, \quad \text{(B6)}$$

where $J \equiv m_j \oint dl |v_\parallel|$ is the parallel or second, adiabatic invariant.

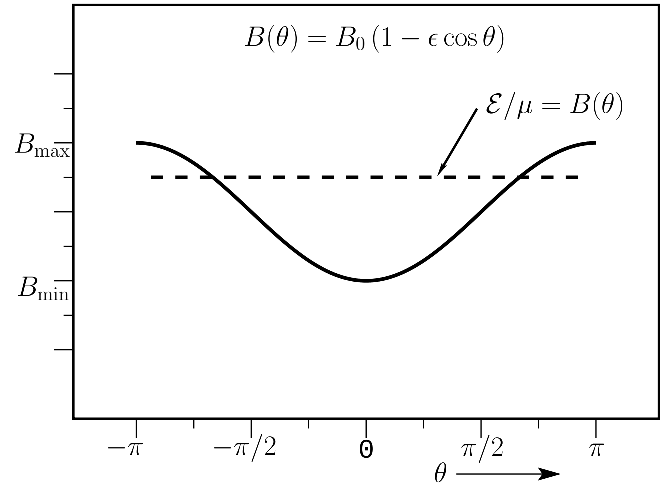

FIG. 6: Sketch of tokamak magnetic mirror field. For a particle with a given kinetic energy $\mathcal{E}$ and magnetic moment μ, the turning points are given by the intersection of the dashed line with $B(\theta)$. For a given particle to be trapped, it must satisfy $\mu B_{\min} \leq \mathcal{E} \leq \mu B_{\max}$.

2. Formal Proof

For lack of a published reference, we shall present a proof of the well-known result that the bounce-averaged guiding center drift is given by (B6) which is based on a derivation by J.D. Callen and C.L. Hedrick.[20] Again using the Clebsh coordinates (Ψ, β, l) we consider the path integral

$$I(\Psi, \beta) = \oint_\gamma \vec{dl} \cdot \vec{v}, \quad \text{(B7)}$$

of a general vector field $\vec{v}$ along a path γ that is tangent to the magnetic field $\vec{B} = \vec{\nabla}\Psi \times \vec{\nabla}\beta$ (see Fig. 5). We wish to calculate the gradient of the line integral given above. Imagine perturbing the coordinate Ψ by adding an infinitesimal change $\Delta\Psi$, which would perturb the path γ to a new path γ' as shown in Fig. 5. The corresponding change in the path integral $I(\Psi, \beta)$ would be

$$\Delta I(\Psi, \beta)_\Psi = \oint_{\gamma'} \vec{dl}' \cdot \vec{v} - \oint_\gamma \vec{dl} \cdot \vec{v}, \quad \text{(B8)}$$

where the coordinates on each line are $(\Psi + \Delta\Psi, \beta, l)$ and (Ψ, β, l) respectively. Using Stoke's theorem, this can be cast in terms of an integral over the area bounded by the two paths which is given by

$$\Delta I_\Psi = \Delta\Psi \oint_\gamma dl\, \vec{\nabla} \times \vec{v} \cdot (\vec{e}_\Psi \times \vec{e}_l), \quad \text{(B9)}$$

where the area vector is $\vec{da} = \Delta\Psi \vec{e}_\Psi \times dl \vec{e}_l$. Dividing by $\Delta\Psi$ and taking the limit $\Delta\Psi \to 0$ gives the partial derivative of I with respect to Ψ:

$$\frac{\partial I}{\partial \Psi} = \oint_\gamma dl\, \vec{\nabla} \times \vec{v} \cdot (\vec{e}_\Psi \times \vec{e}_l). \quad \text{(B10)}$$

Thus, we find the gradient of the path integral (B7) is

$$\vec{\nabla} I = \vec{\nabla}\Psi \oint_\gamma dl\, \vec{\nabla}\times\vec{v}\cdot(\vec{e}_\Psi \times \vec{e}_l) + \vec{\nabla}\beta \oint_\gamma dl\, \vec{\nabla}\times\vec{v}\cdot(\vec{e}_\beta \times \vec{e}_l). \tag{B11}$$

For the special case that $\vec{v}$ is parallel to $\vec{B}$, the curl of $\vec{v}$ reduces to $\vec{\nabla}\times\vec{v} = \vec{\nabla}S \times \vec{e}_l + S\vec{\nabla}\times\vec{e}_l$. Calculating the component of the curl in the direction of $\vec{da}$ depicted in Fig. 5 we find

$$\vec{\nabla}\times\vec{v}\cdot(\vec{e}_\Psi\times\vec{e}_l) = \vec{e}_{\Psi\perp}\cdot\vec{\nabla}S - S\,\vec{e}_\Psi\cdot\vec{\kappa}, \tag{B12}$$

where $\vec{e}_{\Psi\perp} \equiv \vec{e}_\Psi - (\vec{e}_\Psi\cdot\vec{e}_l)\vec{e}_l$ and the magnetic field curvature is $\kappa \equiv \vec{e}_l\cdot\vec{\nabla}\vec{e}_l$. (In general $\vec{\nabla}l$ and $\vec{e}_l$ do not point in the same direction, and thus $\vec{e}_{\Psi\perp} \neq \vec{e}_\Psi$.) An identical relation holds for $\vec{\nabla}\times\vec{v}\cdot(\vec{e}_\beta\times\vec{e}_l)$. Recalling that $\vec{e}_j, \vec{e}^{\,i}$ form a complimentary set of base vectors, we may choose to express the contravariant base vectors in terms of the covariant base vectors; for example

$$\vec{\nabla}\Psi = B\vec{e}_\beta\times\vec{e}_l. \tag{B13}$$

With this in mind, straight forward algebra yields

$$\frac{\vec{\nabla}I\times\vec{B}}{B^2} = \vec{e}_{\Psi\perp}\oint dl\left[\vec{e}_{\beta\perp}\cdot\vec{\nabla}S - S\,\vec{e}_\beta\cdot\vec{\kappa}\right] - \vec{e}_{\beta\perp}\oint dl\left[\vec{e}_{\Psi\perp}\cdot\vec{\nabla}S - S\,\vec{e}_\Psi\cdot\vec{\kappa}\right], \tag{B14}$$

and the proof is almost complete. Since a particle's energy is conserved, the guiding center drift (A1) can be written

$$\vec{v}_{dj} = \frac{m_j v_\parallel}{Z_j e}\vec{e}_l\times\left[-\frac{\vec{\nabla}v_\parallel}{B} + v_\parallel\frac{\vec{\kappa}}{B}\right]. \tag{B15}$$

The first term is

$$-\frac{\vec{e}_l\times\vec{\nabla}v_\parallel}{B} = -\vec{e}_l\times\left[\vec{e}_\Psi\cdot\vec{\nabla}v_\parallel\frac{\vec{\nabla}\Psi}{B}\right. \left. \vec{e}_\beta\cdot\vec{\nabla}v_\parallel\frac{\vec{\nabla}\beta}{B} + \vec{e}_l\cdot\vec{\nabla}v_\parallel\frac{\vec{\nabla}l}{B}\right], \tag{B16}$$

which by expanding the contravariant base vectors in terms of the covariant set as in (B13) can be reduced to

$$-\frac{\vec{e}_l\times\vec{\nabla}v_\parallel}{B} = \vec{e}_{\beta\perp}\cdot\vec{\nabla}v_\parallel\,\vec{e}_{\Psi\perp} - \vec{e}_{\Psi\perp}\cdot\vec{\nabla}v_\parallel\,\vec{e}_{\beta\perp}. \tag{B17}$$

Using similar logic, the curvature drift term in (B15) is

$$\frac{\vec{e}_l\times\vec{\kappa}}{B} = \vec{e}_\Psi\cdot\vec{\kappa}\,\vec{e}_{\beta\perp} - \vec{e}_\beta\cdot\vec{\kappa}\,\vec{e}_{\Psi\perp}, \tag{B18}$$

and thus the guiding-center drift (B15) becomes

$$\vec{v}_{dj} = \frac{m_j v_\parallel}{Z_j e}\left[\vec{e}_{\beta\perp}\cdot\vec{\nabla}v_\parallel - v_\parallel\vec{e}_\beta\cdot\vec{\kappa}\right]\vec{e}_{\Psi\perp} - \frac{m_j v_\parallel}{Z_j e}\left[\vec{e}_{\Psi\perp}\cdot\vec{\nabla}v_\parallel - v_\parallel\vec{e}_\Psi\cdot\vec{\kappa}\right]\vec{e}_{\beta\perp}. \tag{B19}$$

We are interested in the bounce-averaged drift in the $\vec{e}_{\Psi\perp}$ and $\vec{e}_{\beta\perp}$ directions and so define the vectorial bounce average as

$$\langle\vec{v}_{dj}\rangle_b \equiv \frac{\vec{e}_{\Psi\perp}}{\tau_b}\oint\frac{dl}{v_\parallel}\vec{v}_{dj}\cdot\vec{\nabla}\Psi + \frac{\vec{e}_{\beta\perp}}{\tau_b}\oint\frac{dl}{v_\parallel}\vec{v}_{dj}\cdot\vec{\nabla}\beta. \tag{B20}$$

Setting $S = m_j v_\parallel$ in (B14) and inspecting (B19) we find immediately that $\langle\vec{v}_{dj}\rangle_b$ is given by (B6), i.e.

$$\langle\vec{v}_{d\,j}\rangle_b = \frac{\vec{\nabla}J\times\vec{B}}{Z_j e\,\tau_b B^2}. \tag{B21}$$

3. A useful identity for the bounce-average drift components

Consider now the straight-field-line coordinates (Ψ,θ,β), where $\beta = q\theta - \zeta$ is a field line label, and $q = \vec{B}\cdot\vec{\nabla}\zeta/\vec{B}\cdot\vec{\nabla}\theta$ is the safety factor. The covariant-base vectors are given by (6)–(8). Using the identity

$$\frac{\partial}{\partial\beta}\left(B|v_\parallel|\right) = \frac{\partial B}{\partial\beta}\left(|v_\parallel| - \frac{\mu B}{m_j|v_\parallel|}\right) = \frac{\partial B}{\partial\beta}\frac{\partial}{\partial\mu}\left(\mu|v_\parallel|\right), \tag{B22}$$

and neglecting any β variation in $\vec{B}\cdot\vec{\nabla}\theta$, the "radial" drift frequency reduces to

$$\dot{\Psi} = \frac{m_j}{Z_j e\,\tau_b}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{\partial B}{\partial\beta}\frac{\partial}{\partial\mu}\left(\mu|v_\parallel|\right). \tag{B23}$$

Using similar logic the "precessional drift" frequency in the $\vec{\nabla}\beta$ direction is

$$\dot{\beta} \simeq \phi'(\Psi) - \frac{m_j}{Z_j e\,\tau_b}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{\partial B}{\partial\Psi}\frac{\partial}{\partial\mu}\left(\mu|v_\parallel|\right), \tag{B24}$$

where the Ψ derivative in $\vec{B}\cdot\vec{\nabla}\theta$ has been neglected.

Appendix C: Derivation of Eq. (32), the $1/\nu$ particle flux

Using the straight field line coordinates (Ψ,θ,β) defined in appendix B and changing the velocity integration to magnetic moment (μ) and kinetic energy ($\mathcal{E}$) phase space, the particle flux is expressed as

$$\Gamma_j = \frac{2\pi}{A\,m_j^2}\sum_\sigma\sigma\int dS\int_0^{\mathcal{E}_c}d\mathcal{E}\int_{\mathcal{E}/B_M}^{\mathcal{E}/B}d\mu\frac{B}{v_\parallel}\,\vec{v}_d\cdot\vec{\nabla}\Psi\,f_j, \tag{C1}$$

where the flux surface integration ($\int dS$) and area (A) are defined in (31). The sum over $\sigma \equiv \mathrm{sgn}(v_\parallel)$ is needed to include all of velocity space in the integration. Interchanging the order of integration between θ and μ [see Figure (7)] gives

$$\int_{-\pi}^{\pi}d\theta\int_{\mathcal{E}/B_M}^{\mathcal{E}/\mathcal{B}}d\mu = \int_{\mathcal{E}/B_M}^{\mathcal{E}/B_m}d\mu\int_{-\theta_c}^{\theta_c}d\theta \tag{C2}$$

where B_M (B_m) is the max (min) of $B(\theta) = B_0(1-\epsilon\cos\theta)$ and θ_c is the turning point defined by $\mu B(\theta_c) = \mathcal{E}$. Interchanging the order of integration in the particle flux yields

$$\Gamma_j = \frac{4\pi}{A\,m_j^2}\int_{-\pi}^{\pi} d\beta \int_0^{\mathcal{E}_c} d\mathcal{E} \oint d\mu\, \tau_b \dot{\Psi} f_j. \tag{C3}$$

Here $\oint d\mu = \int_{\mathcal{E}/B_M}^{\mathcal{E}/B_m} d\mu$, and the "radial" drift frequency $\dot{\Psi}$ is given by (B23).

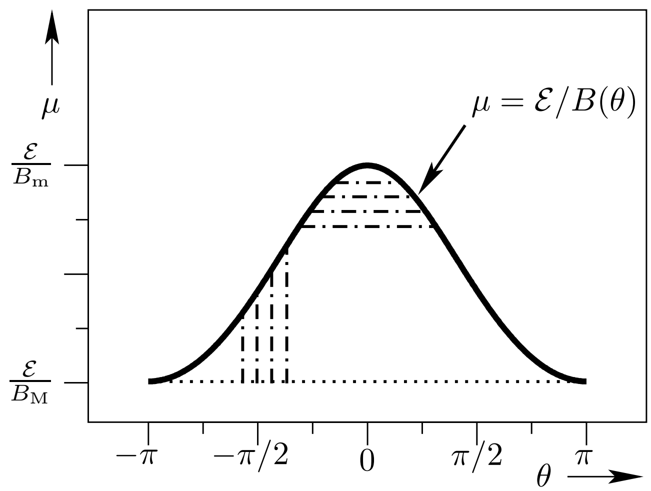

FIG. 7: Illustration of integration region in μ–θ space for the model toroidal magnetic mirror field $B(\theta) = B_0(1-\epsilon\cos\theta)$ employed in this paper. Vertical dashed-dotted lines denote Riemann subintervals for performing μ integration first: $\int_{-\pi}^{+\pi} d\theta \int_{\mathcal{E}/B_{\rm M}}^{\mathcal{E}/B(\theta)} d\mu$. Horizontal dashed-dotted lines denote Riemann subintervals for performing θ integration first: $\int_{\mathcal{E}/B_{\rm M}}^{\mathcal{E}/B_{\rm m}} d\mu \int_{-\theta_c}^{+\theta_c} d\theta$, where the turning point is $\theta_c = \cos^{-1}([\lambda-1]/\lambda\epsilon)$, with $\lambda = \mu B_0/\mathcal{E}$.

We proceed in the standard fashion [16] by noting that the collision frequency is much larger than the "radial drift" frequency (B23), and expand $f = f_0 + f_1 + \cdots$ in powers of $\delta \equiv \dot{\Psi}/[\nu_D^j \Delta\Psi]$ assuming for the moment that $\nu_D^j \sim \dot{\beta}/\Delta\beta$. The equilibrium distribution contributes a term

$$\Gamma_0 \propto \int_{-\pi}^{\pi} d\beta\, \tau_b \dot{\Psi} f_0(\Psi) \propto \int_{-\pi}^{\pi} d\beta \frac{\partial}{\partial\beta} J = 0, \tag{C4}$$

which is zero since the integral is a total derivative in β by virtue of (B4). Thus, the lowest order particle flux is

$$\Gamma_j = \frac{4\pi}{A\,m_j^2}\int_{-\pi}^{\pi} d\beta \int_0^{\mathcal{E}_c} d\mathcal{E} \oint d\mu\, \tau_b \dot{\Psi}\, f_1; \tag{C5}$$

i.e., it is first order in δ. Inserting (27) and integrating by parts once in μ we find

$$\Gamma_{1/\nu\, j} = -\frac{4\pi}{A\,m_j}\int_{-\pi}^{\pi} d\beta \int_0^{\mathcal{E}_c} \frac{\nu_D^j}{f_0'(\Psi)} d\mathcal{E} \times \oint \mu\, d\mu \left(\oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}|v_\parallel|\right)\left(\frac{\partial f_1}{\partial\mu}\right)^2. \tag{C6}$$

(Note that the boundary term at the turning points is zero since the distribution function for trapped particles is zero there.) Solving for $\partial f_{j,(1,0)}/\partial\mu$ in (28) and calculating $\partial B/\partial\beta$ from (11) we have

$$\Gamma_{1/\nu\, j} = -\frac{4\pi^2 (B_0)^2}{A\,m_j (Z_j e)^2}\int_0^{\mathcal{E}_c} \frac{f_0'(\Psi) v(\mathcal{E})}{\nu_D^j} d\mathcal{E} \times \oint \frac{\mu d\mu}{\left(\oint d\theta |v_\parallel|/[v\vec{B}\cdot\vec{\nabla}\theta]\right)} \sum_n n^2 \left[\tilde{a}_n^2 + \tilde{b}_n^2\right], \tag{C7}$$

where

$$\tilde{a}_n = \oint \frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{|v_\parallel|}{v} A_n(\theta), \tag{C8}$$

and likewise for $\tilde{b}_n$ with A_n replaced with B_n. Finally, changing the integration from magnetic moment μ to normalized pitch angle κ defined in (35) gives to lowest order

$$\oint \sqrt{\lambda}\,\mu\, d\mu \simeq \frac{2\,\mathcal{E}^2\,\epsilon}{(B_0)^2}\int_0^1 d\kappa^2. \tag{C9}$$

The magnitude of the parallel velocity expressed in terms of κ is

$$\frac{|v_{\parallel,0}|}{v} = \sqrt{2\lambda\epsilon}\sqrt{\kappa^2 - \sin^2(\theta/2)}. \tag{C10}$$

After some algebra, the particle flux in the $1/\nu$ regime reduces to

$$\Gamma_{1/\nu\, j} = -\frac{\epsilon^{3/2} I_\lambda}{m_j^{3/2}(Z_j e)^2}\int_0^{\mathcal{E}_c} \frac{f_0'(\Psi)\mathcal{E}^{5/2}}{\nu_D^j} d\mathcal{E}, \tag{C11}$$

which is identically (32).

Appendix D: Derivation of Eq. (58), the ν regime particle flux

For the case when $1 \le \delta_2$, we employ a subsidiary expansion for $f_{j,1}$ using $1/\delta_2$ or

$$\delta_3 = \frac{\nu_D^j \Delta\beta}{\dot{\beta}} \ll 1, \tag{D1}$$

as a small parameter. The lowest order equation

$$\dot{\Psi}\frac{\partial f_{j,(0)}}{\partial\Psi} + \dot{\beta}\frac{\partial f_{j,(1,0)}}{\partial\beta} = 0, \tag{D2}$$

can be rewritten using (B23) and (B24) to yield

$$\frac{\partial f_{j,(1,0)}}{\partial\beta} \simeq \frac{f_0'(\Psi)m_j}{Z_j e\,\phi'(\Psi)\tau_b}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\left[-\frac{\partial B}{\partial\beta}\right]\frac{\partial}{\partial\mu}\left(\mu|v_\parallel|\right). \tag{D3}$$

Note that the $\partial/\partial\beta$ operator driving the non-ambipolar flux [see (D8) below] operates on the magnetic field amplitude alone. This, combined with the fact that we are interested in toroidally-trapped banana particles allows us to approximate $|v_\parallel| \simeq |v_{\parallel 0}|$ in the bounce integral above. Using (11) and (12) we integrate once in β to find

$$f_{1,0} = \frac{f_0'(\Psi)m_j}{Z_j e\,\phi'(\Psi)\tau_b}\oint\frac{d\theta\,B}{\vec{B}\cdot\vec{\nabla}\theta}\sum_n D_n(\theta,\beta)|v_{\parallel 0}| \times\left[1-\frac{\mu B}{m_j|v_{\parallel 0}|^2}\right] + g(\Psi;\mathcal{E},\mu). \tag{D4}$$

Here $g(\Psi;\mathcal{E},\mu)$ is a constant of integration to be determined from the next higher order equation

$$\dot{\beta}\frac{\partial f_{1,1}}{\partial\beta} = \frac{\nu_D^j}{\tau_b}\frac{\partial}{\partial\mu}\left[\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}m_j\,|v_{\parallel 0}|\,\mu\,\frac{\partial f_{1,0}}{\partial\mu}\right]. \tag{D5}$$

Integrating once in β yields

$$\nu_D^j\frac{\partial}{\partial\mu}\left[\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}m_j\,|v_{\parallel 0}|\,\mu\,\frac{\partial g}{\partial\mu}\right] = 0, \tag{D6}$$

indicating $\partial g/\partial\mu = 0$. Since the non-ambipolar particle flux is proportional to $\partial f_{1,0}/\partial\mu$ [see (D9) below] we may neglect $g(\Psi;\mathcal{E})$ entirely in what follows. From (C5) the particle flux in the ν regime is

$$\Gamma_{\nu j} = \frac{4\pi}{A\,m_j^2}\int_{-\pi}^{\pi}d\beta\int_{\mathcal{E}_c}^{\infty}d\mathcal{E}\oint d\mu\,\tau_b\dot{\Psi}\left(f_{1,0}+f_{1,1}\right). \tag{D7}$$

Using (D2), the term involving $f_{1,0}$ is expressible as a total derivative in β and vanishes. The remaining term is

$$\Gamma_{\nu j} = -\frac{4\pi}{A\,m_j^2}\int_{\mathcal{E}_c}^{\infty}\frac{d\mathcal{E}}{f_0'(\Psi)}\int_{-\pi}^{\pi}d\beta\oint d\mu\,\tau_b\dot{\beta}\frac{\partial f_{10}}{\partial\beta}f_{1,1}. \tag{D8}$$

Note that via (D3) if $\partial B/\partial\beta \to 0$ then $\Gamma_\nu \to 0$; thus returning to axisymmetry makes the non-ambipolar particle flux vanish as we would expect. Integrating once by parts in β, using (D5) to replace $\partial f_{1,1}/\partial\beta$ in favor of the collision operator on $f_{1,0}$ and integrating once more by parts in μ yields

$$\Gamma_{\nu j} = -\frac{4\pi}{A\,m_j}\int_{\mathcal{E}_c}^{\infty}\frac{\nu_D^j}{f_0'(\Psi)}d\mathcal{E}\int_{-\pi}^{\pi}d\beta \times\oint\mu\,d\mu\left(\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}|v_{\parallel 0}|\right)\left(\frac{\partial f_{1,0}}{\partial\mu}\right)^2, \tag{D9}$$

which is identical to (C6). Now, using (D4) and performing the integration in β to diagonalize the (squared) series in n, we find the ν-regime flux is given by

$$\Gamma_{\nu j} = -\frac{4\pi^2 m_j}{A\,[Z_j e\phi'(\Psi)]^2}\int_{\mathcal{E}_c}^{\infty}\nu_D^j f_0'(\Psi)d\mathcal{E}\oint\mu\,d\mu \times\left(\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}|v_{\parallel 0}|\right)\sum_n\left[\tilde{\alpha}_n^2+\tilde{\beta}_n^2\right]. \tag{D10}$$

The bounce integrals over the magnetic field harmonics are defined by

$$\tilde{\alpha}_n \equiv \frac{\partial}{\partial\mu}\left(\frac{1}{\tau_b}\oint\frac{d\theta B}{\vec{B}\cdot\vec{\nabla}\theta}\left[|v_\parallel| - \frac{\mu B}{m_j|v_\parallel|}\right]A_n(\theta)\right), \tag{D11}$$

$$\tilde{\beta}_n \equiv \frac{\partial}{\partial\mu}\left(\frac{1}{\tau_b}\oint\frac{d\theta B}{\vec{B}\cdot\vec{\nabla}\theta}\left[|v_\parallel| - \frac{\mu B}{m_j|v_\parallel|}\right]B_n(\theta)\right). \tag{D12}$$

Changing variables from μ to the κ defined in (35), the above bounce integrals reduce to

$$\tilde{\alpha}_n = \frac{\partial}{\partial\mu}\left(\frac{v^2}{\hat{\tau}_b}\oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{BA_n(\theta)}{\sqrt{\kappa^2-\sin^2(\theta/2)}} \times\left[2\lambda\epsilon\left(\kappa^2-\sin^2(\theta/2)\right)-\frac{\lambda}{2}\left(1-\epsilon\cos(\theta)\right)\right]\right), \tag{D13}$$

and likewise for $\tilde{\beta}_n$. The dimensionless bounce time is defined by

$$\hat{\tau}_b = \oint\frac{d\theta}{\vec{B}\cdot\vec{\nabla}\theta}\frac{B}{\sqrt{\kappa^2-\sin^2(\theta/2)}} \tag{D14}$$

$$= \sqrt{\frac{4\epsilon\lambda\mathcal{E}}{m_j}}\,\tau_b. \tag{D15}$$

Straightforward algebra on the terms in square brackets appearing in (D13) yields [...] $= 2\lambda\epsilon\left[3/2\kappa^2 - 3/2\sin^2(\theta/2)\right] - 1/2 = 3/2\left|v_{\parallel,0}/v\right|^2 - 1/2$. Thus we find

$$\tilde{\alpha}_n = \frac{\partial}{\partial\mu}\left(\frac{v^2}{2\hat{\tau}_b}\oint\frac{d\theta\,B}{\vec{B}\cdot\vec{\nabla}\theta}\frac{\left[3\left|v_{\parallel,0}/v\right|^2-1\right]}{\sqrt{\kappa^2-\sin^2(\theta/2)}}A_n(\theta)\right), \tag{D16}$$

and likewise for $\tilde{\beta}_n$. Incorporating all of this into (D13) gives a compact form for the particle flux:

$$\Gamma_{\nu j} = -\frac{G_\lambda\,\epsilon^{-1/2}}{m_j^{3/2}[Z_j e\phi'(\Psi)]^2}\int_{\mathcal{E}_c}^{\infty}\nu_D^j\,f_0'\,\mathcal{E}^{5/2}\,d\mathcal{E}, \tag{D17}$$

where G_λ is defined in (60). Finally, inserting the deflection frequency from (22) gives (58).

The normalized bounce time (D14) can only be reduced further with knowledge of $\vec{B}\cdot\vec{\nabla}\theta$ [which is related to the Jacobian, J, of the (Ψ, θ, ζ) straight-field-line coordinates via $J^{-1} = \vec{B}\cdot\vec{\nabla}\theta]$. However if $\vec{B}\cdot\vec{\nabla}\theta$ has no θ dependence, (D14) becomes

$$\hat{\tau}_b = \frac{2B_0\,(1-\epsilon)}{\vec{B}\cdot\vec{\nabla}\theta}\int_0^{\theta_c}\frac{d\theta}{\hat{v}_\parallel} + \frac{4B_0\epsilon}{\vec{B}\cdot\vec{\nabla}\theta}\int_0^{\theta_c}\frac{\sin^2(\theta/2)\,d\theta}{\hat{v}_\parallel}, \tag{D18}$$

where $\hat{v}_\parallel = \sqrt{\kappa^2 - \sin^2(\theta/2)}$, and the turning point is $\theta_c = 2\sin^{-1}\kappa$. These integrals can be reduced to complete elliptic integrals via the substitution $u = \sin(\theta/2)/\kappa$ to yield

$$\hat{\tau}_b = \frac{4B_0}{\vec{B}\cdot\vec{\nabla}\theta}\left[(1+\epsilon)K(\kappa) - 2\epsilon\,E(\kappa)\right]. \tag{D19}$$

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