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The absence of complete FLR stabilization in extended MHD

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Abstract

In the framework of extended MHD, the finite Larmor radius (FLR) effects due to ion gyroviscosity and generalized Ohm's law are well known to stabilize the pure interchange g mode. However, recent theory and simulation indicate that, the complete FLR stabilization of the pure interchange g mode, may not be attainable by the ion gyroviscosity or the two-fluid effect alone in the framework of extended MHD, for a class of plasma equilibria in certain finite-β or non-isentropic regimes.

It is well known that the kinetic effects due to finite Larmor radius (FLR) are able to stabilize the pure interchange mode in a weakly unstable plasma under gravity [1]. The dominant FLR stabilization effects on the gravitational instability (g mode) can be retained by taking into account of the ion gyroviscosity or the generalized Ohm's law in an extended MHD model [2]. Recently Ferraro and Jardin [3] extended earlier work of Roberts and Taylor [2] by including effects of plasma compression. They found the FLR effects due to ion gyroviscosity alone can completely stabilize the g mode of the isothermal equilibrium they considered in all plasma β regimes.

Extended MHD model has been widely applied in the simulation studies of edge localized modes (ELMs) in tokamaks. The FLR stabilization of interchange-like, high-n ballooning modes in extended MHD provides a natural cut-off of the high-n spectrum without resort to numerical or artificial dissipation for ELM simulations. Direct benchmark between theory and extended MHD codes for the linear ballooning instabilities in ELMs has not been conclusive due to the complexity of the edge tokamak plasma equilibrium involved [4]. On the other hand, the FLR stabilization of g mode may provide a simpler case for benchmark between theory and codes, while serving a paradigm for the FLR stabilization in more complicated situcations, as suggested by Schnack *et al.* [5].

In a recent code verification effort, Schnack and Kruger [6] computed the linear growth of a g mode using the NIMROD code with the implementation of the extended MHD model [7]. For the particular equilibrium considered, however, they didn't find the complete FLR stabilization as predicted by the earlier theories that were based on extended MHD model [2, 3]. In an effort to resolve the discrepancy, we revisited the analytical dispersion relation of the pure interchange g mode, in the model of compressible extended MHD, for a general shearless slab configuration.

Consider the following extended MHD model in a cartesian coordinate system as in [2]

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{u} \tag{1}$$

$$\frac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B} + \rho\mathbf{g} - \nabla\cdot\boldsymbol{\pi}_i \tag{2}$$

$$\frac{dp}{dt} = -\gamma p\nabla\cdot\mathbf{u} \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \tag{4}$$

$$\mu_0\mathbf{J} = \nabla\times\mathbf{B} \tag{5}$$

$$\mathbf{E} = -\mathbf{u}\times\mathbf{B} + \frac{\lambda}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e\right) \tag{6}$$

$$(\pi_i)_{xx} = -(\pi_i)_{yy} = -\frac{\delta p_i}{2\Omega}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \tag{7}$$

$$(\pi_i)_{xy} = (\pi_i)_{yx} = \frac{\delta p_i}{2\Omega}\left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) \tag{8}$$

where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$, γ is the adiabatic index, $\mathbf{g}$ the gravity, n the number density, p_i (p_e) the ion (electron) pressure, p the total pressure ($p = p_i + p_e$), Ω the ion gyrofrequency, and the rest of symbols are conventional. We also use two multipliers λ and δ to track the two-fluid and the ion gyroviscosity effects, respectively.

A two-fluid static equilibrium is specfied as follows:

$$\mathbf{u} = 0 \tag{9}$$

$$\mathbf{B} = B\mathbf{e}_z \tag{10}$$

$$\frac{d}{dx}\left(p + \frac{B^2}{2}\right) = \rho\mathbf{g}\cdot\mathbf{e}_x = \rho g \tag{11}$$

$$ne\mathbf{E} = \nabla p_i - \rho\mathbf{g}. \tag{12}$$

The equilibrium is assumed to vary only in x, and $\mathbf{e}_i (i = x, y, z)$ is the basis vector in each cartesian direction. The pure interchange perturbation for g mode has the form

$$\mathbf{u} = [u_x(x)\mathbf{e}_x + u_y(x)\mathbf{e}_y]e^{ik_y y - i\omega t} \tag{13}$$

and it satisfies the local approximation ordering: $k_y L_x \sim \epsilon$, $k_y d_i \sim \lambda \sim \delta$, $u_y \sim \epsilon u_x$, $\epsilon \ll 1$, where $L_x = |d\ln A/dx|^{-1}$ is the spatial scale of field A in x direction, and $d_i = \sqrt{p_i/\rho}/\Omega$ is the ion Larmor radius.

When both the ion gyroviscous tensor and the generalized Ohm's law are kept in the extended MHD model, to the lowest order in ϵ we obtain the local dispersion relation for the pure interchange g mode

$$\omega(\omega^2 + \omega_*\omega + \gamma_{\mathrm{FLR}}^2) + D = 0 \tag{14}$$

where

$$\omega_* = \frac{k_y}{\Omega}\frac{\delta\tau\left[(1+\gamma\beta)(1+\beta)\frac{p'}{\rho} - (2+\gamma\beta)g\beta\right] - \lambda\left[g - \tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)' + \frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p^2}{\rho^2}\frac{\rho'}{\rho}\right]}{(1+\gamma\beta)\left(1 + \frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}\right)} \tag{15}$$

$$
\begin{aligned}
\gamma_{\rm FLR}^2 &= \gamma_{\rm GYR}^2 \\
&+ \frac{k_y^2\lambda\delta\tau}{\Omega^2}\frac{p}{\rho}\frac{(1+\beta)\left(\tau\frac{p'}{\rho}-g\right)\frac{p'}{p}+\left[(1+\gamma\beta\tau)g-(1+\beta)\gamma\tau\frac{p'}{\rho}\right]\frac{\rho'}{\rho}+\left(\frac{\rho g}{p}-\tau\frac{p'}{p}\right)g\beta}{(1+\gamma\beta)\left(1+\frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}\right)}
\end{aligned} \tag{16}
$$

$$
\gamma_{\rm GYR}^2 = \frac{\gamma_{\rm MHD}^2}{1+\frac{k_y^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}} \tag{17}
$$

$$
\gamma_{\rm MHD}^2 = \frac{g^2}{u_{\rm A}^2(1+\gamma\beta)}-\frac{\rho'}{\rho}g \tag{18}
$$

$$
D = -\frac{k_y\lambda}{\Omega}\frac{\frac{\rho'}{\rho}g\tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)'}{(1+\gamma\beta)\left(1+\frac{k_y^2\delta^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}\right)} \tag{19}
$$

Here, $\Omega = eB/m_i$ (with m_i being ion mass), $\beta = \mu_0 p/B^2$, $u_{\rm A}^2 = B^2/\mu_0\rho$, $\tau = p_i/p$, and $A' = dA/dx$. In obtaining Eq.(14), no assumption about the ordering of ω is explicitly made. This dispersion relation is applicable in finite-β and general non-isentropic thermal plasma regimes. It recovers the dispersion relation in [2] for an isentropic plasma with zero β. Whereas the extended MHD model is only valid in small Larmor radius regime where $(\lambda,\delta) \ll 1$, the above dispersion relation is formally applicable in regimes where $(\lambda,\delta) \sim 1$ as well. Such a feature allows the benchmarking of extend MHD simulations in a wide range of parameter regimes. Earlier a similar dispersion relation was obtained by Ferraro and Jardin [3] for an isothermal plasma in the low frequency regime. The dispersion above applies to general plasma equilibrium, including the isothermal case. However, as shown later, different equilibrium can bring out qualitative difference in FLR stabilization.

Formally setting $\lambda = 0, \delta = 1$, the local dispersion relation for the pure interchange g mode reduces to the following form with FLR effects due only to ion gyroviscous force in momentum equation

$$
\omega^2 + \omega_*\omega + \gamma_{\rm GYR}^2 = 0 \tag{20}
$$

where

$$
\omega_* = \frac{\frac{k_y\tau}{\Omega}\left[(1+\beta)\frac{p'}{\rho}-\frac{2+\gamma\beta}{1+\gamma\beta}g\beta\right]}{1+\frac{k_y^2\tau^2}{4\Omega^2}\frac{p}{\rho}\frac{\beta}{1+\gamma\beta}} \tag{21}
$$

Complete FLR stabilization requires

$$
\omega_*^2 > 4\gamma_{\rm GYR}^2, \tag{22}
$$

$$\text{or} \qquad \frac{k_y^2\tau^2}{\Omega^2} \geq \frac{k_c^2\tau^2}{\Omega^2} = \frac{4\gamma_{\rm MHD}^2}{\left[(1+\beta)\frac{p'}{\rho} - \frac{2+\gamma\beta}{1+\gamma\beta}g\beta\right]^2 - \frac{p}{\rho}\frac{\beta}{1+\gamma\beta}\gamma_{\rm MHD}^2} \tag{23}$$

where k_c is the cut-off wavenumber. When $\beta \to 0$, there is always a k_c for complete FLR stabilization by ion gyroviscosity, which reduces to the case shown in [2]. For finite β, there is a possibility that a real value of k_c may not exist because of the negative sign in front of the second term in the denominator of the expression for k_c in (23). For the isothermal plasma equilibrium studied by Ferraro and Jardin [3], $p' = p\rho'/\rho$, and the cut-off wavenumber $k_c^{\rm FJ}$ is given by

$$\left(\frac{k_c^{\rm FJ}\tau\beta}{\Omega}\right)^2 = \frac{4\gamma_{\rm MHD}^2}{\left[\frac{u_{\rm A}^2}{L_\rho}(1+\beta) + \frac{2+\gamma\beta}{1+\gamma\beta}g\right]^2 - \frac{u_{\rm A}^2}{1+\gamma\beta}\gamma_{\rm MHD}^2} \tag{24}$$

where $L_\rho = -(d\ln\rho/dx)^{-1}$ which is assumed to be independent of β. It can be shown that for $g/L_\rho > 0$, the right hand of (24) is constantly positive, therefore a real value of $k_c^{\rm FJ}$, hence the complete FLR stabilization of the g mode, can be always found for the isothermal equilibrium in all regimes of β. This is consistent with the findings in [3].

For the equilibrium where the magnetic field is uniform, as was studied by Schnack and Kruger [6], $p' = \rho g$, so that the cut-off wavenumber $k_c^{\rm SK}$ is determined by

$$\begin{aligned}\left(\frac{k_c^{\rm SK}\tau}{\Omega}\right)^2 &= \frac{4\gamma_{\rm MHD}^2}{g^2\left(1-\frac{\beta}{1+\gamma\beta}\right)^2 - \frac{u_{\rm A}^2\beta^2}{1+\gamma\beta}\gamma_{\rm MHD}^2} \\ &= \frac{4(1+\gamma\beta)\gamma_{\rm MHD}^2}{g^2[1+(\gamma-2)\beta] - u_{\rm A}^2\frac{g}{L_\rho}\beta^2} \\ &= \frac{4(1+\gamma\beta)\gamma_{\rm MHD}^2}{u_{\rm A}^2\frac{g}{L_\rho}(\beta_- - \beta)(\beta_+ + \beta)}\end{aligned} \tag{25}$$

where

$$\beta_\pm = \frac{\sqrt{(2-\gamma)^2 g^4 + \frac{4u_{\rm A}^2 g^3}{L_\rho}} \pm (2-\gamma)g^2}{2u_{\rm A}^2\frac{g}{L_\rho}} \tag{26}$$

When $g/L_\rho > 0$ the denominator in the right hand side of (25) is a monotonically decreasing function of β (for $\beta > 0$), and becomes zero and negative when $\beta \geq \beta_{\rm crit} = \beta_-$. As it turns out, for the particular equilibrium case studied by Schnack and Kruger [6] in NIMROD simulations, $\beta_{\rm crit} \sim 0.45$, whereas in the center of the simulation domain $\beta \sim 0.5$.

For the same equilibrium, we further performed a more detailed comparison between the NIMROD simulations and the dispersion relation in (20). The results are shown in

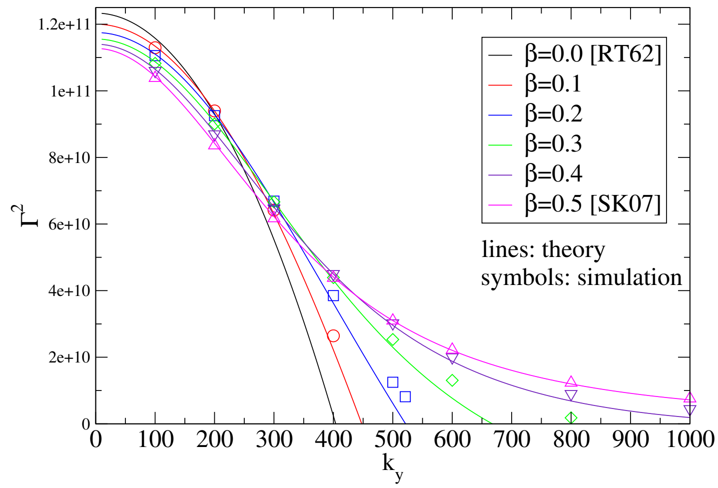

FIG. 1: Comparison between theory and NIMROD simulations for the g mode dispersion relation.

Fig. 1. The square of the linear growth growth rate (Γ^2) are plotted as a function of k_y for equilibria with an increasing set of β values (where β is represented by the value measured at the center of the simulation domain). The analytical dispersions are plotted as lines, whereas the symbols are values measured from simulations. Both analytical and simulational data for the same equilibrium (as defined by each β value) are plotted with same color. The black line is analytical dispersion relation for zero β which is same as that given by Roberts and Taylor [2]. There is no simulation data for this case, since in simulation zero pressure brings in numerical difficulty. The magenta color ($\beta = 0.5$) represents the case originally studied by Schnack and Kruger [6]. This case is qualitatively different from all other cases in the figure, in that $\beta > \beta_{\rm crit}$ and there is no cut-off k_y at all. Theory and simulation have a reasonable agreement except very near marginal stability. In that situation, the growth rate and the cut-off wavenumber obtained from simulations tend to be larger than the analytic values. In passing it is noted that the plotted dispersion also indicates the existence of a critical $k_y \sim 300$; below this wavenumber, the growth rate decreases with β, and above the value, the growth rate increases with β for any fixed k_y.

In the case when only the two-fluid effects are included by the generalized Ohm's law whereas the gyroviscosity is ignored, the local dispersion relation for the pure interchange g mode can be obtained as follows by formally setting $\lambda = 1, \delta = 0$ in (14):

$$\omega(\omega^2 + \omega_* \omega + \gamma_{\rm MHD}^2) + D = 0 \tag{27}$$

where

$$\omega_* = -\frac{k_y}{\Omega}\frac{1}{1+\gamma\beta}\left[g-\tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)'\right] = -\frac{k_y}{\Omega}\frac{1}{1+\gamma\beta}\left[g+\tau\left(c_s^2\frac{\rho'}{\rho}-\frac{p'}{\rho}\right)\right] \quad (28)$$

$$D = -\frac{k_y}{\Omega}\frac{\frac{\rho'}{\rho}g}{1+\gamma\beta}\tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)' = \frac{k_y}{\Omega}\frac{\frac{\rho'}{\rho}g}{1+\gamma\beta}\tau\left(c_s^2\frac{\rho'}{\rho}-\frac{p'}{\rho}\right) \quad (29)$$

and $c_s^2 = \gamma p/\rho$, $\tau = p_i/p$, with p_i being the ion pressure. The above dispersion relation reduces to that in [2] in an isentropic plasma where the entropy density $\ln(p/\rho^\gamma)$ is a constant. For non-isentropic plasma where $D \neq 0$, there are 3 eigenmodes. When $g/L_\rho/\Omega$ or D is not very small, there are situations when there are 2 complex conjugate roots so that there's always one growing mode for any k_y. When that happens, the complete FLR stabilization could be absent.

In the low frequency or weakly unstable regime, where $(g/L_\rho)/(\omega\Omega) \ll 1$ so that $D \sim 0$, the complete FLR stabilization criterion is simply $\omega_*^2 > 4\gamma_{\rm MHD}^2$, or

$$\frac{k_y^2}{\Omega^2} \geq \frac{k_c^2}{\Omega^2} = \frac{4(1+\gamma\beta)^2\gamma_{\rm MHD}^2}{\left[g-\tau\frac{p}{\rho}\left(\ln\frac{p}{\rho^\gamma}\right)'\right]^2} \quad (30)$$

Again it is also possible to find an equilibrium such that the denominator in the expression for cut-off wavenumber in (30) becomes identical or close to zero, so that the complete FLR stabilization effects could be entirely lost.

For example, for the equilibrium with uniform magnetic field, the cut-off wavenumber has the form

$$\frac{k_c^2}{\Omega^2} = \frac{4(1+\gamma\beta)^2\gamma_{\rm MHD}^2}{\left[\frac{\tau\gamma u_{\rm A}^2}{L_\rho}(\beta-\beta_{\rm crit})\right]^2} \quad (31)$$

where $\beta_{\rm crit} = (1-\tau)gL_\rho/(\tau\gamma u_{\rm A}^2)$. Similarly, for the isothermal equilibrium,

$$\frac{k_c^2}{\Omega^2} = \frac{4(1+\gamma\beta)^2\gamma_{\rm MHD}^2}{\left[\frac{\tau(\gamma-1)u_{\rm A}^2}{L_\rho}(\beta-\beta_{\rm crit})\right]^2} \quad (32)$$

where $\beta_{\rm crit} = gL_\rho/[\tau(\gamma-1)u_{\rm A}^2]$. However unlike the case where the FLR stabilization is due to ion gyroviscosity alone, the absence of complete FLR stabilization here strictly only occurs at rather limited regime of $\beta \sim \beta_{\rm crit}$.

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