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Neoclassical toroidal viscosity and error-field penetration in tokamaks

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Abstract

A model for field error penetration is developed that includes non-resonant as well as the usual resonant field error effects. The non-resonant components cause a neoclassical toroidal viscous torque that tries to keep the plasma rotating at a rate comparable to the ion diamagnetic frequency. The new theory is used to examine resonant error-field penetration threshold scaling in ohmic tokamak plasmas. Compared to previous theoretical results, the plasma is found to be *less* susceptible to error-field penetration and locking, by a factor that depends on the non-resonant error-field amplitude.

1 Introduction

Efforts to understand the penetration of non-axisymmetric magnetic field perturbations—"error-fields"—into high temperature plasmas have concentrated on the role of resonant components. Resonant helical magnetic perturbations are those whose wave vector $\vec{k}$ is perpendicular to the equilibrium magnetic field somewhere inside the plasma i.e., where $\vec{k} \cdot \vec{B}_0 = 0$ (which is equivalent to the statement $q = m/n$). Such a surface is termed the "resonant surface" for the particular mode in question. In this work, it is shown that non-resonant magnetic field perturbations can play a crucial role in the error-field penetration problem by producing a global neoclassical torque that damps toroidal flow toward a diamagnetic ion-type flow. In contrast, a resonant perturbation produces a localized electromagnetic braking torque at its respective resonant surface. Accounting for both these effects leads to a criterion for the error-field penetration which indicates that the critical resonant error-field amplitude increases with plasma density, a result that is in qualitative agreement with empirical scaling [1].

2 History and Motivation

Considerable theoretical [2–6] and experimental [1, 7–12] effort has been aimed at understanding the effects of small resonant helical magnetic field errors—arising from field coil misalignments and non-axisymmetric coil feed-throughs—on plasma confinement in tokamaks. The impetus for this research has come from the experimental correlation between the emergence of locked modes and disruptions in tearing-stable low-density ohmic discharges. Error-field locked modes are induced and develop as follows [1, 7]: 1) the resonant error field is ramped up linearly or the electron density is ramped down slowly (> 100 ms), 2) when the locked mode threshold is reached, a rapid (~ 5 ms) bifurcation to a non-rotating "locked-state" is observed, and then 3) for ~ 100 ms a stationary magnetic island—driven by the error field—develops (usually on the $q = 2$ surface) and leads to either a major disruption or confinement degradation. Locked mode avoidance in low-density ohmic discharges is highly desirable—if not crucial—for reliable tokamak operation.

To date, the theoretical and experimental error-field studies have been confined to predicting the resonant (e.g., $m/n = 2/1$) critical error-field strength (as a function of plasma density, toroidal field strength, and other variables) when bifurcation occurs and after which a locked mode develops. Currently, empirical and theoretical locked mode thresholds do not agree on the scaling to larger devices. Predictive capability for locked-mode avoidance on ITER [13] is needed. The present benchmark scenario for ITER relies on an ohmic start-up with an anticipated low toroidal rotation rate (~ 0.5 kHz).

3 Conventional Error-field Theory

The standard model [4–6, 14] employed to describe error-field penetration considers the response of a large aspect ratio toroidally-rotating tearing-stable plasma to a single resonant helical magnetic perturbation. The plasma is approximated by a periodic cylinder, with nearly circular flux surfaces. Standard cylindrical coordinates (r, θ, z) and simulated toroidal coordinates (r, θ, ϕ) with $z = R_0\phi$ will be employed in this work. The resonant field component exerts an electromagnetic torque on the plasma only in the vicinity of its rational surface [4]. This torque is brought about by the nonlinear interaction of error-field-induced eddy-currents in a singular layer around the rational surface with the error-field itself and is directed against the flow, trying to brake the plasma. Theoretical predictions of the eddy current response in the layer depend on the physics model employed. The standard model assumes a flow drive plus a phenomenological diffusive perpendicular viscous torque that opposes the electromagnetic braking torque, trying to maintain the plasma flow profile. The steady-state balance between electromagnetic and viscous torques yields a transcendental equation whose roots give the modified layer velocity (in the presence of the resonant error-field) as a function of error-field strength. Above a critical error-field strength the electromagnetic torque on the resonant surface exceeds the perpendicular viscous torque on the plasma flow, and the rational surface bifurcates to a stationary, or *locked* state. This bifurcation is termed *error-field penetration*, and the critical error-field strength at which it occurs is known as the *penetration threshold*. After locking, magnetic reconnection on the resonant surface proceeds unhindered, as if there were no equilibrium plasma flow. This scenario closely mimics observations of error-field penetration occurring during the ohmic start-up phase of several tokamaks [1, 7–11].

4 Neoclassical Toroidal Viscosity

While resonant components of the magnetic field perturbation spectrum have dominated the theoretical discussion, in tokamak experiments many non-resonant components are also present. While the non-resonant components in and of themselves cannot produce locking, these components can effect the plasma through their role in damping the toroidal flow by a neoclassical viscous torque mechanism. Recent experiments on NSTX with large applied non-resonant magnetic perturbations demonstrated qualitative and quantitative agreement [15] with theoretical predictions [16] of toroidal flow damping.

4.1 Origin of neoclassical toroidal viscosity

In the context of fluid theory, Neoclassical Toroidal Viscosity [NTV] can be understood as the toroidal drag force experienced by the plasma moving along distorted flux surfaces having broken toroidal symmetry. Within a kinetic neoclassical context [16–18] it can be thought of as being induced by collision and $\vec{E} \times \vec{B}$ moderated radial particle drifts that cause a non-ambipolar radial particle flux. We will consider the drag induced by an error field consisting of one resonant (i.e., $m = 2$, $n = 1$) and many non-resonant harmonics. Assuming the error-field-induced distortion within the toroidal plasma is small enough that the flux surface remains intact on average, we may employ the theoretical formulation of Shaing [16–18]. On each flux surface, the magnetic field strength is decomposed into helical harmonics in Hamada coordinates (Θ, ζ) by

$$B = B_0 (1 - \epsilon \cos \Theta) + \sum_{(n,m)\neq(0,0)} \left[b_{nmc} \cos (m\Theta - n\zeta) + b_{nms} \sin (m\Theta - n\zeta) \right], \qquad (1)$$

where $\epsilon = r/R_0$, r is the minor radial coordinate, and R_0 is the major radius of the magnetic axis. In the above the b_{nmc}, b_{nms} coefficients are effectively the "shielded" magnetic perturbations inside the plasma. However, we assume the contribution from resonant harmonics to the total NTV force (17) is small. Henceforth, we will assume that the coefficients b_{nmc}, b_{nms} are to a good approximation given by their values in vacuum. The toroidal momentum dissipation arising from

NTV is described through the toroidal component of the ion viscous stress tensor and leads to a toroidal flow velocity evolution equation of the form [16]

$$\partial_t \left\langle \vec{e}_\phi \cdot \vec{V} \right\rangle = -\left\langle (1/\rho_m)\vec{e}_\phi \cdot \vec{\nabla} \cdot \overleftrightarrow{\Pi} \right\rangle + \cdots , \tag{2}$$

where ρ_m is the mass density, $\vec{e}_\phi$ is the covariant basis vector pointing in the toroidal direction, $\overleftrightarrow{\Pi}$ is the ion viscous stress tensor, and $\langle \cdots \rangle$ denotes a flux surface average. The precise form of the NTV force depends upon the collisionality of the plasma. For parameters of interest to present day tokamaks, the low collision frequency $1/\nu$ regime, or the low collision frequency ν regime [16] are likely to be the most applicable.

4.2 Evaluating the NTV force in the low collisionality ($1/\nu$) regime

When $q\omega_E < \nu_i/\epsilon < \sqrt{\epsilon}\,\omega_{ti}$, where $\omega_E = E_r/(rB)$ is the poloidal $\vec{E} \times \vec{B}$ drift frequency and $\omega_{ti} \equiv v_{ti}/(R_0 q)$ is the ion transit frequency with $V_{ti} \equiv (T_i/m_i)^{1/2}$, the toroidally trapped particles (bananas) are “collisionless,” and dominate the non-ambipolar diffusion process. In this collision frequency limit ion neoclassical toroidal viscosity dominates by a factor of order $\sqrt{m_i/m_e}$ over the electron NTV. In the large aspect ratio limit, the $1/\nu$ regime ion NTV force is given by [16]

$$F_\phi^{NC,1/\nu} = -\left\langle (1/\rho_m)\vec{e}_\phi \cdot \vec{\nabla} \cdot \overleftrightarrow{\Pi} \right\rangle = -\nu_{\|,1/\nu} \left[b_{1/\nu}(r)\right]^2 \left[V_\phi(r) - V_{*,1/\nu}^{NC}(r)\right] , \tag{3}$$

where

$$\left[b_{1/\nu}(r)\right]^2 \simeq 1.74\, q^2 \epsilon^{3/2} \sum_n \sum_{m,m'} n^2 \frac{(b_{nmc}b_{nm'c} + b_{nms}b_{nm's})}{B_0^2} B_{\lambda,1/\nu}, \tag{4}$$

$$B_{\lambda,1/\nu} \equiv \int_0^1 d\kappa^2 \frac{F_{nmc}(\kappa)F_{nm'c}(\kappa)}{\mathrm{E}(\kappa) - (1-\kappa^2)\mathrm{K}(\kappa)}, \tag{5}$$

and

$$F_{nmc}(\kappa) \equiv \oint d\Theta \sqrt{\kappa^2 - \sin^2(\Theta/2)}\, \cos\left[(m - nq)\Theta\right]. \tag{6}$$

Here $b_{1/\nu}(r)$ is an effective measure of the magnetic perturbation level $\delta B/B$ in $|B|$ in the $1/\nu$ regime. The complete elliptic integrals of the first and second kind, $K(k)$ and $E(k)$, are defined with the k^2 convention:

$$K(k) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}, \tag{7}$$

$$E(k) \equiv \int_0^{\pi/2} d\theta \sqrt{1 - k^2 \sin^2\theta}. \tag{8}$$

In the above $V_\phi \equiv \vec{e}_\phi \cdot \vec{V}$ is the toroidal flow speed, $\nu_{\|,1/\nu} = \omega_{ti}^2/\nu_i$, ν_i is the ion-ion collision frequency, κ is a normalized pitch-angle variable defined in [16], and $\oint d\Theta \equiv \int_{-\Theta_b}^{+\Theta_b}$ (the lowest order toroidal equilibrium is assumed to have symmetric bounce points). The neoclassical toroidal flow velocity for the $1/\nu$ regime is [16]

$$V_{*,1/\nu}^{NC}(r) \simeq \frac{3.5}{Z_i e\, B_\theta} \frac{dT_i}{dr}, \tag{9}$$

where $Z_i e$ is the charge of the ion species. This neoclassical flow velocity is in a direction *counter* to the plasma current, I_p.

4.3 Evaluating the NTV force in the low collisionality (ν) regime

As the collisionality continues to decrease, the radial drift of ions becomes oscillatory and is limited to $\Delta r_D \sim V_{Dr}/(q\omega_E)$ by the toroidal $\vec{E} \times \vec{B}$ precession drift. Then, collisions induce radial transport with a diffusivity coefficient $D \propto \nu_{\rm eff}(\Delta r_D)^2 \sim (\nu_i/\epsilon)V_{Dr}^2/(q\omega_E)^2$. Though it is unlikely that present day tokamak ion temperatures are high enough to be deeply in the ν regime [16], it is possible that the ions are *slightly* inside the ν regime, i.e. $\nu_i/\epsilon \lesssim q\omega_E$ while the electrons are still well into the $1/\nu$ regime, with $q\omega_E < \nu_e/\epsilon < \sqrt{\epsilon}\,\omega_{te}$. In this limit, ion NTV still dominates over electron NTV. In this case, the large aspect ratio limit of the ion NTV force takes the form [16]

$$F_\phi^{NC,\nu} = -\nu_{\parallel,\nu}\ [b_\nu(r)]^2 \left[V_\phi(r) - V_{*,\nu}^{NC}(r)\right], \tag{10}$$

where $\nu_{\parallel,\nu} = \nu_i \omega_{ti}^2/\omega_E^2$,

$$[b_\nu(r)]^2 \simeq 1.44\epsilon^{-1/2} \sum_n \sum_{m,m'} \frac{(b_{nmc}b_{nm'c} + b_{nms}b_{nm's})}{B_0^2} B_{\lambda,\nu} \tag{11}$$

with

$$B_{\lambda,\nu} \equiv \int_0^1 d\kappa^2 \left[E(\kappa) - (1-\kappa^2)K(\kappa)\right] \frac{\partial L_{nmc}}{\partial \kappa^2}\frac{\partial L_{nm'c}}{\partial \kappa^2}, \tag{12}$$

$$L_{nmc} \equiv \frac{1}{K(\kappa)} \oint d\Theta \left[3\epsilon(\kappa^2 - \sin^2(\Theta/2)) - \frac{1}{2}\right] \frac{\cos\left[(m-nq)\Theta\right]}{\sqrt{\kappa^2 - \sin^2(\Theta/2)}}. \tag{13}$$

Here $b_\nu(r)$ is an effective measure of the magnetic perturbation level $\delta B/B$ in $|B|$ in the ν regime. The neoclassical toroidal flow velocity for the ν regime is [16]

$$V_{*,\nu}^{NC}(r) \simeq \frac{0.92}{Z_i e\, B_\theta}\frac{dT_i}{dr}. \tag{14}$$

Equations (3) and (10) describe the ion dominant NTV force in the two asymptotic collisionality regimes. In what follows, both limiting cases are explored, together with the relevant drift-MHD layer regimes (to be discussed) to calculate new error-field penetration thresholds when ion dominant NTV is present in the vicinity of the resonant surface.

5 Steady-state Toroidal Flow Profile

In the large aspect ratio limit, a toroidal plasma may be approximated by a periodic cylinder, with nearly circular flux surfaces. Standard cylindrical coordinates (r,θ,z) and simulated toroidal coordinates (r,θ,ϕ) with $z = R_0\phi$ will be employed in this work. In the following, dimensionless quantities are employed with all length scales normalized to r_s, the resonant-surface minor radius. The major and minor radii of the plasma are R_0 and a (normalized to r_s), respectively. The magnetic field is normalized to $B_l \equiv s(r_s)B_\theta(r_s)$, where $s(r_s) = (d\ln q/d\ln r)_{r_s}$ represents the magnetic shear at the resonant surface. Here, $q(r) = rB_0/R_0B_\theta(r)$ is the safety-factor profile. All time scales are normalized to $\tau_l = (r_s/V_l)$, where $V_l = B_l/\sqrt{\mu_0\rho_m(r_s)}$, and $\rho_m(r_s)$ is the mass density at the resonant surface.

The equilibrium toroidal momentum balance equation in the absence of error-fields is:

$$\frac{1}{r}\frac{d}{dr}\left[\mu(r)r\frac{dV_\phi^0}{dr}\right] = -F_0. \tag{15}$$

Its solution,

$$V_\phi^0(r) = V_0\left[\int_1^a \frac{xdx}{\mu(x)}\right]^{-1}\int_r^a \frac{xdx}{\mu(x)}, \tag{16}$$

satisfies the boundary conditions $V_\phi(a) = 0$ and $V_\phi(1) = V_0$. Here, $\mu(r)$ is the (phenomenological) ion perpendicular viscosity [normalized to $V_l r_s \rho_m(r_s)$] that represents cross-field momentum transport due to collisional effects or microturbulence. The driving force $F_0 = 2V_0[\int_1^a xdx/\mu(x)]^{-1}$ supports the flow against perpendicular viscous damping with the boundary at $r = a$.

In the presence of static error fields, two additional forces enter the toroidal momentum balance equation. The first—a resonant electromagnetic torque—is strongly localized around the resonant surface and can be represented by $F_{EM}\delta(r-1)/r$, where $\delta(r-1)$ is the Dirac delta function. (The coefficient F_{EM} must be resolved using boundary layer analysis on the resonant surface and will be specified in what follows.) The second force arises from NTV (discussed above) and is of the general form:

$$F_\phi^{NC} = -\nu_\parallel\tau_l\, b^2(r)\left[V_\phi(r) - V_*^{NC}(r)\right]. \tag{17}$$

The effective perturbed magnetic field profile $b^2(r)$ is given by either (4) in the $1/\nu$ regime or (11) in the ν regime. The effective parallel damping rate $\nu_\parallel$, is $\nu_{\parallel,1/\nu} = \omega_{ti}^2/\nu_i$ in the $1/\nu$ regime, or $\nu_{\parallel,\nu} = \nu_i\omega_{ti}^2/\omega_E^2$ in the ν regime. Thus, the new toroidal momentum balance equation is given by

$$\frac{1}{r}\frac{d}{dr}\left(\hat{\mu}(r)r\frac{dV_\phi(r)}{dr}\right) - \hat{b}^2(r)\Gamma_s^2\left[V_\phi(r) - V_*^{NC}(r)\right] = -\frac{F_{E,M}}{\mu_s}\frac{\delta(r-1)}{r} - \frac{F_0}{\mu_s}, \tag{18}$$

where $\hat{\mu} = \mu(r)/\mu_s$, $\mu_s = \mu(r_s)$, $\hat{b}(r) = b(r)/b(r_s)$, and

$$\Gamma_s = \sqrt{\nu_\parallel\tau_l/\mu_s}\, b(r_s). \tag{19}$$

The parameter Γ_s determines whether perpendicular (anomalous or collisional) viscosity dominates over parallel (neoclassical toroidal) viscosity [NTV] in the bulk plasma. In the limit $\Gamma_s \ll 1$, NTV is negligible and the previous drift-MHD theory is obtained [6]. In the opposite limit $\Gamma_s \gg 1$, NTV dominates over perpendicular viscosity, and a WKB-type [19] Green function can be used to find the solution of (18) above [20]:

$$V_\phi(r) \simeq \left[V - V_*^{NC}(1)\right]\frac{\exp\left(-\Gamma_s|1-r|\right)}{\sqrt{r\hat{b}(r)\sqrt{\hat{\mu}(r)}}} + V_*^{NC}(r). \tag{20}$$

Here, $V_*^{NC}(1)$ is given by either (9) or (14) evaluated at the resonant surface $\hat{r} = r/r_s = 1$ (and normalized to V_l described above), depending on whether NTV is in the $1/\nu$ or ν low-collisionality regimes, respectively.

6 Resonant Surface Torque Balance

The error-field penetration threshold is obtained by integrating the toroidal torques across the resonant surface [5] (i.e., $\int\int\int r\,dr\,dz\,d\theta R_0\,\{\ (18)\ \}$). Inspection of (18) and (20) reveals that the neoclassical layer torque ($T_{\phi,NTV}$) and perpendicular viscous torque ($T_{\phi,VS}$) satisfy $T_{\phi,NTV} \simeq \delta\,\Gamma_s\,T_{\phi,VS}$, where $\delta \ll 1$ is the linear layer thickness. We assume

$$1 \ll \Gamma_s \ll 1/\delta, \tag{21}$$

which guarantees NTV may be neglected within the resonant layer, but dominates perpendicular viscosity in the bulk plasma. This constraint has two consequences: (i) as in previous drift-MHD work [6], the resonant layer toroidal torque balance expression is still between (albeit modified) perpendicular viscous and electromagnetic torques [i.e., $T_{\phi,VS} + T_{\phi,EM} = 0$]; and (ii) we can use the previous drift-MHD analysis [6] to evaluate the plasma response in the resonant layer.

The layer response function is given by $\Delta = \partial \ln[b_r(r)]/\partial r|_{1-}^{1+}$. For consistency with layer results in [6], we define the Lundquist number as $S = \tau_R/\tau_H$, where $\tau_R = \mu_0 r_s^2/\eta(r_s)$ and $\tau_H = (R_0\sqrt{\mu_0\rho_m(r_s)})/\,[ns(r_s)B_\phi] = \tau_l/m$. Here $\eta(r_s)$ is the (dimensional) parallel neoclassical resistivity at the resonant surface. Similar to [6] we define dimensionless frequencies $Q = S^{1/3}\omega\tau_H$, $Q_*^{NC} \sim [R_0 m/(r_s n)]S^{1/3}\omega_{*,i}\,\tau_H$, and a scaled plasma response parameter $\hat{\Delta} = S^{-1/3}\Delta$. Here $\omega = mV_{\theta,0}/r_s - nV/R_0$ is the (dimensional) resonant surface frequency in the presence of resonant and non-resonant error-fields, and $\omega_{*,i}$ is the (dimensional) ion diamagnetic flow frequency at the resonant surface. The new steady-state torque balance equation for the resonant layer ($T_{\phi,EM} + T_{\phi\,VS} = 0$) when $\Gamma_s \gg 1$ is [20]

$$\left|\frac{b_r^{\rm vac}}{B_\phi}\right|^2 \frac{{\rm Im}\{\hat{\Delta}(Q)\}}{|\alpha + \hat{\Delta}(Q)|^2} = \frac{P}{\kappa S}(Q_*^{NC} - Q), \tag{22}$$

where $\kappa \equiv 1/([s(r_s)]^2\Gamma_s)$, and $b_r^{\rm vac}$ is the *vacuum* radial magnetic perturbation associated with the resonant error-field component (at the resonant surface). As in [6] $\alpha \equiv -S^{-1/3}\Delta_s'$ is the (normalized to $S^{-1/3}$) conventional tearing stability index of the (stable) m, n mode, $P \equiv \tau_R/\tau_V = \mu_0\mu_i(r_s)/[\eta(r_s)\rho_m(r_s)]$ is the magnetic Prandtl number at the resonant surface, and the perpendicular viscous timescale is given by $\tau_V = r_s^2\rho_m(r_s)/\mu_i(r_s)$, where $\mu_i(r_s)$ is the (dimensional) viscosity. Since $S \gg 1$ and $P \geq 1$ in a high temperature tokamak plasma and a tearing-stable m, n mode is assumed, $|\Delta_s'| \sim \mathcal{O}(1)$, $\alpha \ll 1$, and thus to a good approximation we may neglect α in the above torque balance equation. The error-field penetration threshold corresponds to the critical error-field amplitude above which torque balance is lost, *i.e.*, where the approximated torque balance equation has no solution [5]. It follows that

$$\left|\frac{b_r^{\rm vac}}{B_\phi}\right|^2_{\rm crit} = \max\left\{\frac{P}{\kappa S}\frac{(Q_*^{NC} - Q)\left|\hat{\Delta}(Q)\right|^2}{{\rm Im}\{\hat{\Delta}(Q)\}}\right\}, \tag{23}$$

where the maximum is obtained by varying Q.

7 Relevant Layer Regimes

Recalling $S \gg 1$, $P \geq 1$, and inspecting Sect. III G of [6], it follows that the three error-field response regimes most applicable to present day tokamaks are the *1st Visco-Resistive* (VRi) regime, the *1st Semi-Collisional* (SCi) regime, and the *1st Hall-Resistive* (HRi) regime—see Table 1 below. The VRi regime holds when $D^2\,P^{1/3} < Q$, the SCi regime holds when $\beta^{1/2}\,D < \sqrt{2}\,Q < \sqrt{2}\,D^2\,P^{1/3}$, and the HRi regime holds when $\sqrt{2}\,Q < \beta^{1/2}\,D$. Here, $\beta = 10\,\mu_0 P_0/(3B_0^2)$ is the toroidal beta, where P_0 is the equilibrium plasma pressure, and $D = S^{1/3}\,\rho_s(r_s)/r_s$. The quantity $\rho_s(r_s)$ is the ion Larmor radius at the resonant surface, calculated using the electron temperature.

Table 1: *Tokamak-relevant linear drift-MHD response regimes [6] for a static error-field. Abbreviations indicate the different response regimes: 1st Hall-Resistive [HRi]; 1st Semi-Collisional [SCi]; 1st Visco-Resistive [VRi]. Here,* $\hat{\Delta} = S^{-1/3}\,\Delta$, $Q = S^{1/3}\,\omega\tau_H$, $Q_{i,e} = -S^{1/3}\,\omega_{*i,e}\,\tau_H$, $D = S^{1/3}\,\rho_s(r_s)/r_s$, *and* $P = \tau_R/\tau_V$. *Here* $\rho_s(r_s)$ *is the ion Larmor radius at the resonant surface, calculated using the electron temperature. Finally,* $\tau = T_i/T_e$ *is the ratio of the ion and electron temperatures. Note the numerical coefficient of the HRi regime differs from that given in [6] owing to a factor of 2 difference in the definition of* β.

Abbreviation	*Response*
HRi	$\hat{\Delta} = 1.786\,[\mathrm{i}\,(Q-Q_e)]\,\beta^{1/4}\,D^{-1/2}\,(1+\tau)^{-1/4}$
SCi	$\hat{\Delta} = 3.142\,[\mathrm{i}\,(Q-Q_i)]^{1/2}\,[\mathrm{i}\,(Q-Q_e)]\,D^{-1}\,(1+\tau)^{-1/2}$
VRi	$\hat{\Delta} = 2.104\,[\mathrm{i}\,(Q-Q_i)]^{1/6}\,[\mathrm{i}\,(Q-Q_e)]^{5/6}\,P^{1/6}$

Using a Padé approximation valid for all values of Γ_s in each of the three layer regimes VRi, SCi, and HRi respectively, we find the error-field penetration threshold in each regime:

$$\left|\frac{b_r^{vac}}{B_\phi}\right|^2_{\text{crit,VRi}} \simeq \frac{[s(r_s)]^2}{\lambda}\frac{P^{7/6}}{S^{1/3}}(\omega_*\tau_H)^2\left[\frac{1+\chi+\chi^2}{1+\chi}\right], \tag{24}$$

$$\left|\frac{b_r^{\text{vac}}}{B_\phi}\right|^2_{\text{crit,SCi}} \simeq \frac{[s(r_s)]^2 r_s}{\lambda R_0}\frac{P(\omega_*\tau_H)^{5/2}}{\rho_* S^{1/2}}\left[\frac{1+\gamma+\gamma^2}{1+\gamma}\right], \tag{25}$$

$$\left|\frac{b_r^{vac}}{B_\phi}\right|^2_{\text{crit,HRi}} \simeq \frac{[s(r_s)]^2\beta^{1/4}}{\rho_*^{1/2}\lambda}\left(\frac{r_s}{R_0}\right)^{1/2}\frac{P}{S^{1/2}}(\omega_*\tau_H)^2\left[\frac{1+\chi+\chi^2}{1+\chi}\right], \tag{26}$$

where $\lambda = 2\int_{r_s}^a [\mu(r_s)/\mu(r)]\,(dr/r)$, $\gamma = [R_0 m/(r_s n)]^{5/2}\lambda\Gamma_s$, and $\chi = [R_0 m/(r_s n)]^2\lambda\Gamma_s$. For simplicity, we have assumed $T_i \simeq T_e$ which implies $\omega_{*,i}$ scales as $\omega_{*,e} \equiv \omega_*$, where $\omega_{*,e}$ is the (dimensional) electron diamagnetic flow frequency at the resonant surface.

In the limit $\Gamma_s \ll 1$, NTV is negligible throughout the plasma and we recover the previous drift-MHD result [6]. Most notably, in the new limit $\Gamma_s \gg 1$ we find that

$$\left|\frac{b_r^{vac}}{B_\phi}\right|^2_{\text{crit,VRi}} \simeq \frac{P^{7/6}}{S^{1/3}}(\omega_*\tau_H)^2\,\Gamma_s, \tag{27}$$

$$\left|\frac{b_r^{\text{vac}}}{B_\phi}\right|^2_{\text{crit,SCi}} \simeq \frac{P(\omega_*\tau_H)^{5/2}}{\rho_* S^{1/2}}\,\Gamma_s, \tag{28}$$

$$\left|\frac{b_r^{vac}}{B_\phi}\right|^2_{\text{crit,HRi}} \simeq \frac{\beta^{1/4}P}{\rho_*^{1/2}S^{1/2}}(\omega_*\tau_H)^2\,\Gamma_s, \tag{29}$$

i.e., the square of the penetration threshold increases by a factor $\sim \Gamma_s \equiv \sqrt{\nu_\| \tau_l/\mu_s}\, b(r_s) = \sqrt{\nu_\| \tau_V}\, b(r_s)$ over the previous results. In this limit, the NTV torque effectively enhances the perpendicular viscosity by reducing the typical bulk velocity profile scale length near the resonant layer, thereby making it more difficult for a resonant field error to lock the rational surface.

8 Tokamak Scaling Study

As an application of this theory, consider a class of ohmically heated tokamak plasmas in which the aspect ratio, R_0/a, and the equilibrium profiles are held *fixed*. By definition, $\omega_*\, \tau_H \propto T_e\, \sqrt{n_e}/\left(R_0\, B_\phi^2\right)$, $S \propto B_\phi\, T_e^{3/2}\, R_0/\sqrt{n_e}$, $\beta \propto n_e T_e/B_\phi^2$, $\rho_* \propto T_e^{1/2}/(R_0\, B_\phi)$, and $P \propto R_0^2\, T_e^{3/2}/\tau_V$.

8.1 Threshold scaling with $1/\nu$ NTV

In addition to the assumptions above, the low collisionality ($1/\nu$) NTV regime sets the effective parallel damping rate as $\nu_{\|,1/\nu} = \omega_{ti}^2/\nu_i \propto T_e^{5/2}/(R_0^2 n_e)$. Thus, in the NTV dominated limit $\Gamma_s \gg 1$, (24)-(26) reduce to

$$\left|\frac{b_r^{\rm vac}}{B_\phi}\right|_{\text{crit,VRi},1/\nu} \sim n_e^{2/3} B_\phi^{-13/3} R_0^{-1} T_e^{9/2} \tau_V^{-2/3}\, \sigma_{NR,\,1/\nu}, \tag{30}$$

$$\left|\frac{b_r^{\rm vac}}{B_\phi}\right|_{\text{crit,SCi-HRi},1/\nu} \sim n_e B_\phi^{-9/2} R_0^{-1} T_e^{4} \tau_V^{-1/2}\, \sigma_{NR,\,1/\nu}, \tag{31}$$

$$\sigma_{NR,\,1/\nu} = \sqrt{\sum_n \sum_{m,m'} n^2 \frac{(b_{nmc} b_{nm'c} + b_{nms} b_{nm's})}{[b_r^{\rm vac}]^2} B_{\lambda,1/\nu}}, \tag{32}$$

where $B_{\lambda,1/\nu}$ is given by (5). (Under the substitutions above, the HRi and SCi regimes scale identically.) Here, $\sigma_{NR,\,1/\nu}$ is the ratio of the "effective" non-resonant to resonant error field at the resonant surface. Ohmic power balance allows us to eliminate T_e in favor of the energy confinement time τ_E:

$$T_e = \left(\frac{\tau_E}{n_e}\right)^{2/5} \left(\frac{B_\phi}{R_0}\right)^{4/5}, \tag{33}$$

which further reduces the penetration thresholds to

$$\left|\frac{b_r^{\rm vac}}{B_\phi}\right|_{\text{crit,VRi},1/\nu} \sim n_e^{2/3} B_\phi^{-11/15} R_0^{-23/5} \left(\frac{\tau_E}{n_e}\right)^{9/5} \tau_V^{-2/3} \sigma_{NR,\,1/\nu}, \tag{34}$$

$$\left|\frac{b_r^{\rm vac}}{B_\phi}\right|_{\text{crit,SCi-HRi},1/\nu} \sim n_e B_\phi^{-13/10} R_0^{-21/5} \left(\frac{\tau_E}{n_e}\right)^{8/5} \tau_V^{-1/2} \sigma_{NR,\,1/\nu}. \tag{35}$$

Experiments on JET, DIII-D, and Alcator C-Mod find the error-field penetration threshold scales approximately linearly with electron density and inversely with toroidal magnetic field strength [1]. Experimental scaling with major radius is not directly measured, but inferred from the observed scalings with electron density and toroidal field strength via dimensionless scaling arguments [1, 21]. More precisely, the empirical penetration threshold is found to scale as $b_r^{\rm vac}/B_\phi =$

$n_e^{\alpha_n} B_\phi^{\alpha_B} R_0^{\alpha_R}$ with $\alpha_n \simeq 1.0$, $-1.2 < \alpha_B < -0.6$ and $0.5 < \alpha_R < 1.25$ [1]. Assuming a neo-Alcator energy confinement scaling $\tau_E \propto n_e R_0^{3.25}$ [22], (34) and (35) simplify to

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,VRi},1/\nu} \sim n_e^{2/3} B_\phi^{-11/15} R_0^{5/4} \tau_V^{-2/3} \sigma_{NR,\,1/\nu}, \tag{36}$$

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,SCi-HRi},1/\nu} \sim n_e B_\phi^{-13/10} R_0 \tau_V^{-1/2} \sigma_{NR,\,1/\nu}. \tag{37}$$

In the above τ_V is left unspecified because $\mu_i(r_s)$ (perpendicular ion viscosity) is unknown and no good theory exists for it in ohmic plasmas.

8.2 Threshold scaling with ν NTV

If instead the plasma is slightly in the ν regime, the effective parallel damping rate is given by $\nu_{\|,\nu} = \nu_i \omega_{ti}^2/\omega_E^2 \propto (R_0^2 B_\phi^2 n_e)/T_e^{5/2}$. (Here it has been assumed that $\omega_E \sim \omega_*$.) Thus, in the NTV dominated limit $\Gamma_s \gg 1$, (24)-(26) reduce to

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,VRi},\nu} \sim n_e^{5/3} B_\phi^{-10/3} R_0 T_e^2 \tau_V^{-2/3}\, \sigma_{NR,\,\nu}, \tag{38}$$

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,SCi-HRi},\nu} \sim n_e^2 B_\phi^{-7/2} R_0 T_e^{3/2} \tau_V^{-1/2}\, \sigma_{NR,\,\nu}, \tag{39}$$

$$\sigma_{NR,\,\nu} = \sqrt{\sum_n \sum_{m,m'} \frac{(b_{nmc} b_{nm'c} + b_{nms} b_{nm's})}{[b_r^{\mathrm{vac}}]^2} B_{\lambda,\nu}}, \tag{40}$$

where $B_{\lambda,\nu}$ is given by (12). (Again under the substitutions above, the HRi and SCi regimes scale identically.) Here, $\sigma_{NR,\,\nu}$ is the ratio of the "effective" non-resonant to resonant error field at the resonant surface in the ν regime. Using the ohmic power balance constraint reduces the above thresholds to

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,VRi},\nu} \sim n_e^{5/3} B_\phi^{-26/15} R_0^{-3/5} \left(\frac{\tau_E}{n_e}\right)^{4/5} \tau_V^{-2/3} \sigma_{NR,\,\nu}, \tag{41}$$

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,SCi-HRi},\nu} \sim n_e^2 B_\phi^{-23/10} R_0^{-1/5} \left(\frac{\tau_E}{n_e}\right)^{3/5} \tau_V^{-1/2} \sigma_{NR,\,\nu}. \tag{42}$$

Again assuming a neo-Alcator energy confinement scaling, we find

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,VRi},\nu} \sim n_e^{5/3} B_\phi^{-26/15} R_0^2 \tau_V^{-2/3} \sigma_{NR,\,\nu}, \tag{43}$$

$$\left|\frac{b_r^{\mathrm{vac}}}{B_\phi}\right|_{\text{crit,SCi-HRi},\nu} \sim n_e^2 B_\phi^{-23/10} R_0^{7/4} \tau_V^{-1/2} \sigma_{NR,\,\nu}. \tag{44}$$

Provided the perpendicular viscous timescale τ_V has no density dependence, the $1/\nu$ regimes (36)-(37) all predict nearly linear density dependence for the penetration threshold—a result that is universally observed in experimental scaling studies [1]. In particular, the $1/\nu$ SCi-HRi regimes (37) capture the linear density scaling exactly. The ν regimes in general have far too strong a density scaling $|b_r^{\rm vac}/B_\phi| \propto n_e^2$ or $n_e^{5/3}$ as compared to experiment (unless τ_V has a strong density dependence). There is considerable uncertainty in the scalings since a neo-Alcator scaling has been assumed for τ_E and τ_V has been left unspecified. Nonetheless the thresholds obtained in both the $1/\nu$ and ν regimes have a stronger dependence on electron density compared to previous theory [6].

9 Conclusions

A new theory for error-field penetration has been developed that accounts for resonant and non-resonant helical magnetic field perturbations. While the non-resonant components cannot induce locking in and of themselves, they can inhibit resonant error-field locking by modifying the plasma flow profile via a neoclassical toroidal viscous [NTV] force. In the limit $1 \ll \Gamma_s \ll 1/\delta$ neoclassical toroidal viscosity [NTV] effectively enhances perpendicular viscosity by reducing the typical bulk velocity profile gradient scale length near (but not within) the resonant layer, thereby making it more difficult for a resonant field error to lock the toroidal flow on the rational surface. The new penetration thresholds all have two novel features: (i) a stronger dependence on electron density than previously predicted [6] (a result in qualitative agreement with empirical scaling studies [1] if (τ_E/n_e) and τ_V do not depend strongly on n_e); (ii) a dependence on the ratio, σ_{NR}, between the non-resonant and resonant error-field components, a feature that could be tested in present tokamaks to determine the relevance of neoclassical toroidal viscosity in ohmic discharges.

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