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# Ballooning filament growth in the intermediate nonlinear regime

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## Abstract

In this work we develop a theory of ballooning instability in the intermediate nonlinear regime for general magnetic configurations including tokamaks. The evolution equations for plasma filament growth induced by the ballooning instability are derived accounting for the dominant nonlinear effects in an ideal MHD description. The intermediate nonlinear regime of ballooning modes is defined by the ordering that the plasma filament length across field line in the direction of pressure gradient is comparable to the mode width in the same direction. In the tokamak case, this regime becomes particularly relevant for a transport barrier as the width of that barrier (or pedestal) region approaches the mode width of the dominant ballooning mode. The filament growth equations in the intermediate nonlinear ballooning regime is meant to quantify the precursor and pre-collapse phase of edge localized modes (ELMs) in both simulations and experiments.

## I. INTRODUCTION

Filamentary structures and their localization in the unfavorable curvature region of the tokamak edge have been routinely observed during periods of edge localized modes (ELMs) in recent MAST experiments [1, 2] and extended MHD simulations [3, 4]. This indicates that the ballooning instability of the pedestal region continues to play a dominant role in determining the nonlinear temporal and spatial structures of ELMs. Thus it may be possible to understand the dynamics of the ELM filaments in terms of the nonlinear properties of ballooning instability.

A typical ELM event in a tokamak would evolve through about three phases with different time scales [2]. The precursor phase has a time scale about $100\mu\text{s} \sim 100 - 1000\tau_A$, where the Alfvén time scale $\tau_A \sim Rq/u_A \sim \sqrt{RL_p}/c_s$, and $R$ is the major radius, $q$ the safty factor, $L_p$ the pedestal width (or pressure gradient scale length), $u_A$ the local Alfvén speed, and $c_s$ the sound speed. Evidence indicates that this phase of ELM is associated with certain linear to early nonlinear MHD activity. The precursor phase terminates at the onset of the collapse phase, during which the edge diagnostic signal levels rapidly increase and drop in a time scale of about $10\mu\text{s} \sim 10 - 100\tau_A$. The recovery phase links the end of the collapse phase to the precursor phase of the next ELM cycle. This phase takes place in a transport time scale $\tau_E \sim 10\text{ms}$ which allows the edge pedestal to recover its height in H-mode.

Different phases of ELMs may relate to different linear and nonlinear regimes of ballooning instability. Introduce two small parameters

$$n^{-1} = \frac{k_\parallel}{k_\perp} \ll 1, \qquad \varepsilon = \frac{|\boldsymbol{\xi}|}{L_{\text{eq}}} \ll 1 \tag{1}$$

where $k_\parallel$ and $k_\perp$ are the dominant wavenumbers of perturbation parallel and perpendicular to equilibrium magnetic field lines, respectively; $\boldsymbol{\xi}$ is the plasma displacement from equilibrium, and $L_{\text{eq}}$ is the equilibrium scale length (which is used as the normalization length later so that $L_{\text{eq}} = 1$). The first small paramenter $n^{-1}$ can be used to measure the three intrinsic spatial scales that partially define the structure of a linear ballooning mode. They are the mode width in the direction parallel to equilibrium magnetic field, $\lambda_\parallel \sim k_\parallel^{-1}$; the mode width in the direction across magnetic flux surface, $\lambda_\wedge$; and the mode width in the direction perpendicular to both of the former two directions, $\lambda_\perp \sim k_\perp^{-1}$. If $L_{\text{eq}}$ is chosen to be the same order of $\lambda_\parallel$, as in $\lambda_\parallel \sim L_{\text{eq}} \sim 1$, then we have $\lambda_\wedge \sim n^{-1/2}$, and $\lambda_\perp \sim n^{-1}$.

The perturbation amplitude of the nonlinear ballooning mode, measured by $\varepsilon$ or $|\boldsymbol{\xi}|$, can be compared to those characteristic spatial scales of its linear mode structure. In the early nonlinear regime, which is defined by the ordering $|\boldsymbol{\xi}| \sim \lambda_\perp \sim n^{-1}$, the filament scale across the magnetic flux surface is comparable to the mode width in the most rapidly oscillating direction [5–7]. In this regime, $|\boldsymbol{\xi}| \ll \lambda_\wedge$, nonlinear convection across the flux surface is small relative to the mode width in that direction, and nonlinearities modify only the mode envelop development across magnetic surface. As the mode continues to grow, it enters the intermediate nonlinear regime, in which $|\boldsymbol{\xi}| \sim \lambda_\wedge \sim n^{-1/2}$, and the plasma displacement across magnetic flux surface becomes of the same order as the mode width in that direction [8, 9]. Convectional and compressional effects are no longer small, and nonlinearities due to convection and compression, together with nonlinear line-bending force, directly modify "local" mode evolution along the magnetic field line. In the late nonlinear regime, the ballooning filament growth may exceed the scale of the pedestal width and result in the collapse of the pedestal. Eventually, these ballooning filaments could detach from edge plasma and propagate into the scrape-off-layer region, as indicated from recent experiment and simulations [2, 4].

It is conceivable that the linear to early nonlinear regime of the ballooning instability of pedestal may mostly correspond to the precursor phase of ELMs since the onset of the ELMs have been consistently correlated to the breaching of linear stability boundry of the peeling-ballooning mode [10, 11]. Earlier theory attempted to explain the collapse onset phase of ELMs by invoking the finite time singularity associated with the early nonlinear ballooning instability of a marginally unstable configuration ("Cowley-Artun regime") [5–7]. Such a scenario, however, has yet to be confirmed by direct MHD simulations, probably due to the rather limited range of that regime. In contrast, there is a good agreement between the prediction of intermediate nonlinear ballooning mode equation and results from direct MHD simulation for the case of a line-tied $g$-mode [8]. It is likely that the intermediate nonlinear regime may better characterize the transition from precursor phase to collapse onset phase of an ELM. This regime could become particularly relevant for a transport barrier as the width of that barrier (or pedestal) region approaches the mode width of the dominant ballooning mode.

To further explore such a scenario, we develop in this work the theory for the intermediate nonlinear ballooning instability in general and toroidal configurations, as the first step. We first lay out the general formulation in Clebsch coordinate in Sec. II, and in a tokamak flux coordinate in Sec. III. To facilitate analytical and numerical analysis, we further consider the circular shaped tokamak in large aspect ratio limit, and formulate our theory in Shafranov coordinates in Sec. IV. The relevant equations for the intermediate nonlinear regime of ballooning instability in an $s-\alpha$ equilibrium are given. Finally we conclude with summary and discussion in Sec. V.

## II. BALLOONING FILAMENT EQUATIONS IN CLEBSCH COORDINATES

The nonlinear theory of ballooning mode can be conveniently developed in the Lagrangian formulation of the ideal MHD model [12].

$$\frac{\rho_0}{J}\nabla_0\mathbf{r}\cdot\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}=-\nabla_0\left[\frac{p_0}{J^\gamma}+\frac{\left(\mathbf{B}_0\cdot\nabla_0\mathbf{r}\right)^2}{2J^2}\right]+\nabla_0\mathbf{r}\cdot\left[\frac{\mathbf{B}_0}{J}\cdot\nabla_0\left(\frac{\mathbf{B}_0}{J}\cdot\nabla_0\mathbf{r}\right)\right]+\frac{\rho_0}{J}\nabla_0\mathbf{r}\cdot\mathbf{g} \quad (2)$$

where

$$\mathbf{r}(\mathbf{r}_0,t)=\mathbf{r}_0+\boldsymbol{\xi}(\mathbf{r}_0,t),\quad \nabla_0=\frac{\partial}{\partial\mathbf{r}_0},\quad J(\mathbf{r}_0,t)=|\nabla_0\mathbf{r}| \quad (3)$$

Here, $\mathbf{r}_0$ denotes the initial location of each plasma element in the equilibrium,and $J(\mathbf{r}_0,t)$ is the Jacobian for the Lagrangian transformation from $\mathbf{r}_0$ to $\mathbf{r}(\mathbf{r}_0,t)$. We first consider a general magnetic configuration

$$\mathbf{B}_0=\nabla_0\Psi_0\times\nabla_0\alpha_0 \quad (4)$$

in a nonorthogonal Clebsch coordinate system $(\Psi_0,\alpha_0,l_0)$, where $\Psi_0$ is the magnetic flux label, $\alpha_0$ the field line label, and $l_0$ the measure of field line length. The plasma filament size due to ballooning instability can be quantified by the plasma displacement $\boldsymbol{\xi}$, which is expanded in terms of $1/n$ and $\varepsilon$ as follows

$$\boldsymbol{\xi}(\sqrt{n}\Psi_0,n\alpha_0,l_0,t)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\varepsilon^i n^{-\frac{j}{2}}\left(\mathbf{e}_\Psi\xi^\Psi_{\{i,j\}}+\frac{\mathbf{e}_\alpha}{\sqrt{n}}\xi^\alpha_{\{i,j\}}+\mathbf{e}_l\xi^l_{\{i,j\}}\right) \quad (5)$$

where $\mathbf{e}_\Psi=B^{-1}\nabla_0\alpha_0\times\nabla_0 l_0$, $\mathbf{e}_\alpha=B^{-1}\nabla_0 l_0\times\nabla_0\Psi_0$, $\mathbf{e}_l=B^{-1}\mathbf{B}=\mathbf{b}$. Here and after we drup the subscript "0" in the equilibrium MHD fields $\rho_0$,$p_0$, and $\mathbf{B}_0$ for convenience. The plasma displacement $\boldsymbol{\xi}$ and the Lagrangian Jacobian $J$ are functions of the normalized coordinates $(\Psi,\alpha,l)$, where $\Psi=\sqrt{n}\Psi_0$, $\alpha=n\alpha_0$, $l=l_0$.

The intermediate nonlinear regime is defined by the ordering $\varepsilon\sim\mathcal{O}(n^{-1/2})$ [8, 9]. In this regime the plasma displacement $\boldsymbol{\xi}$ and the Lagrangian Jacobian $J$ are expanded as a single series of $1/n$

$$\boldsymbol{\xi}(\sqrt{n}\Psi_0, n\alpha_0, l_0, t) \;=\; \sum_{j=1}^{\infty} n^{-\frac{j}{2}} \left( \mathbf{e}_\Psi \xi^\Psi_{\frac{j}{2}} + \frac{\mathbf{e}_\alpha}{\sqrt{n}} \xi^\alpha_{\frac{j+1}{2}} + \mathbf{b}\xi^l_{\frac{j}{2}} \right) \tag{6}$$

$$J(\sqrt{n}\Psi_0, n\alpha_0, l_0, t) \;=\; 1 + J_0 + \sum_{j=1}^{\infty} n^{-\frac{j}{2}} J_{\frac{j}{2}} \tag{7}$$

Setting $\boldsymbol{\xi} = 0$ gives the equilibrium relation

$$\nabla_0 (p + \frac{B^2}{2}) = \mathbf{B} \cdot \nabla_0 \mathbf{B} + \rho \mathbf{g} \tag{8}$$

The lowest order expansions of the full MHD equation (2) yield the equations

$$\mathcal{O}(\sqrt{n}): \quad \left[ \frac{B^2}{(1+J_0)^3} + \frac{\gamma p}{(1+J_0)^{\gamma+1}} \right] \partial_\Psi J_0 = 0 \tag{9}$$

$$\mathcal{O}(n): \quad \left[ \frac{B^2}{(1+J_0)^3} + \frac{\gamma p}{(1+J_0)^{\gamma+1}} \right] \partial_\alpha J_0 = 0 \tag{10}$$

$$\mathcal{O}(1): \quad -p\partial_l \frac{1}{(1+J_0)^\gamma} + \left[ \frac{1}{1+J_0} - \frac{1}{(1+J_0)^\gamma} \right] \partial_l p = 0 \tag{11}$$

This indicates, to leading order, the incompressible condition, $J_0 = const = 0$.

For convenience, a new set of basis vectors are introduced [5]

$$\mathbf{e}_\perp = \mathbf{e}_\Psi \cdot (\mathbf{I} - \mathbf{bb}), \qquad \mathbf{e}_\wedge = \mathbf{e}_\alpha \cdot (\mathbf{I} - \mathbf{bb}) \tag{12}$$

so that

$$\boldsymbol{\xi} = \mathbf{e}_\perp \xi^\Psi + \mathbf{e}_\wedge \xi^\alpha + \mathbf{B}\xi^\parallel \tag{13}$$

At next order, perpendicular and parallel force balance yield, respectively

$$(\gamma p + B^2) J_{\frac{1}{2}} - \mathbf{B} \cdot \nabla_0 (\mathbf{B} \cdot \boldsymbol{\xi}_{\frac{1}{2}}) + 2\mathbf{B} \cdot \nabla_0 \mathbf{B} \cdot \boldsymbol{\xi}_{\frac{1}{2}} + \rho \mathbf{g} \cdot \boldsymbol{\xi}_{\frac{1}{2}} = F_{\frac{1}{2}}(l,t) \tag{14}$$

$$\rho B \partial_t^2 \xi^\parallel_{\frac{1}{2}} = \partial_l (\gamma p J_{\frac{1}{2}}) - \rho \mathbf{g} \cdot \mathbf{b} J_{\frac{1}{2}} + \rho \mathbf{g} \cdot \partial_l \boldsymbol{\xi}_{\frac{1}{2}} \tag{15}$$

With the generalized natural boundary condition $F_{\frac{1}{2}}(l,t) = 0$, we have

$$J_{\frac{1}{2}} = \frac{B^{-2}}{1+\gamma\beta} \left[ B^3 \partial_l \xi^\parallel_{\frac{1}{2}} - \rho \mathbf{g} \cdot \mathbf{B} \xi^\parallel_{\frac{1}{2}} - \mathbf{e}_\perp \cdot (2B^2 \boldsymbol{\kappa} + \rho \mathbf{g}) \xi^\Psi_{\frac{1}{2}} \right] \tag{16}$$

From the parallel force balance (15) we therefore obtain the lowest order equation for $\xi^\parallel$ which remains linear formally [5]

$$\rho B^2 \partial_t^2 \xi^\parallel_{\frac{1}{2}} \;=\; \mathcal{L}_\parallel (\xi^\Psi_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) \tag{17}$$

$$
\begin{aligned}
\mathcal{L}_\parallel(\xi^\Psi_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) \;=\;& B\partial_l\left(\frac{\gamma p}{1+\gamma\beta}B\partial_l\xi^\parallel_{\frac{1}{2}}\right) \\
&+\left[\frac{(\rho\mathbf{g}\cdot\mathbf{b})^2}{1+\gamma\beta}+\rho B\partial_l(\mathbf{g}\cdot\mathbf{B})-B\partial_l\left(\frac{\gamma\beta}{1+\gamma\beta}\rho\mathbf{g}\cdot\mathbf{B}\right)\right]\xi^\parallel_{\frac{1}{2}} \\
&-B\partial_l\left(\mathbf{e}_\perp\cdot\frac{2\gamma p\boldsymbol{\kappa}-\rho\mathbf{g}}{1+\gamma\beta}\xi^\Psi_{\frac{1}{2}}\right) \\
&+\left[\frac{\rho\mathbf{g}\cdot\mathbf{B}}{\gamma p+B^2}\mathbf{e}_\perp\cdot(2B^2\boldsymbol{\kappa}+\rho\mathbf{g})-\mathbf{e}_\perp\cdot\mathbf{g}B\partial_l\rho\right]\xi^\Psi_{\frac{1}{2}} \qquad (18)
\end{aligned}
$$

At the third order, nonlinearity starts to enter the equation for $\xi^\Psi$:

$$
\begin{aligned}
&\rho\left\{|\mathbf{e}_\perp|^2\partial_\alpha\partial_t^2\xi^\Psi_{\frac{1}{2}}+|\mathbf{e}_\perp|^2[\xi^\Psi_{\frac{1}{2}},\partial_t^2\xi^\Psi_{\frac{1}{2}}]+B^2[\xi^\parallel_{\frac{1}{2}},\partial_t^2\xi^\parallel_{\frac{1}{2}}]\right\} \qquad (19)\\
&\quad=\partial_\alpha\mathcal{L}_\Psi(\xi^\Psi_{\frac{1}{2}},\xi^\parallel_{\frac{1}{2}})+[\boldsymbol{\xi}_{\frac{1}{2}};B\partial_l(B\partial_l\boldsymbol{\xi}_{\frac{1}{2}})-\mathbf{B}B\partial_l J_{\frac{1}{2}}-(\rho\mathbf{g}+2B\partial_l\mathbf{B})J_{\frac{1}{2}}]
\end{aligned}
$$

where $[A,B]\equiv\partial_\Psi A\partial_\alpha B-\partial_\alpha A\partial_\Psi B$, $[\mathbf{A};\mathbf{B}]\equiv\partial_\Psi\mathbf{A}\cdot\partial_\alpha\mathbf{B}-\partial_\alpha\mathbf{A}\cdot\partial_\Psi\mathbf{B}$, and

$$
\begin{aligned}
\mathcal{L}_\Psi(\xi^\Psi_{\frac{1}{2}},\xi^\parallel_{\frac{1}{2}}) \;=\;& B\partial_l(|\mathbf{e}_\perp|^2B\partial_l\xi^\Psi_{\frac{1}{2}})+\left[\frac{(\rho\mathbf{g}\cdot\mathbf{e}_\perp)^2}{B^2(1+\gamma\beta)}+\frac{4\mathbf{e}_\perp\cdot\boldsymbol{\kappa}}{1+\gamma\beta}\mathbf{e}_\perp\cdot(\rho\mathbf{g}-\gamma p\boldsymbol{\kappa})\right. \\
&\left.+2\mathbf{e}_\perp\cdot\boldsymbol{\kappa}\mathbf{e}_\perp\cdot(\nabla_0 p-\rho\mathbf{g})-\mathbf{e}_\perp\cdot\mathbf{g}\mathbf{e}_\perp\cdot\nabla_0\rho\right]\xi^\Psi_{\frac{1}{2}} \\
&+\frac{\mathbf{e}_\perp\cdot(2\gamma p\boldsymbol{\kappa}-\rho\mathbf{g})}{1+\gamma\beta}B\partial_l\xi^\parallel_{\frac{1}{2}} \\
&+\left[\frac{\rho\mathbf{g}\cdot\mathbf{B}}{B^2(1+\gamma\beta)}\mathbf{e}_\perp\cdot(2B^2\boldsymbol{\kappa}+\rho\mathbf{g})-\mathbf{e}_\perp\cdot\mathbf{g}\mathbf{B}\cdot\nabla_0\rho\right]\xi^\parallel_{\frac{1}{2}} \qquad (20)
\end{aligned}
$$

The linear operators $\mathcal{L}_\Psi$ and $\mathcal{L}_\parallel$ are same as those obtained earlier in [5]. The governing equations for the filament growth due to ballooning instability in the intermediate nonlinear regime ($\varepsilon\sim n^{-1/2}$) are therefore obtained as follows

$$
\begin{aligned}
&\rho|\mathbf{e}_\perp|^2(\partial_\alpha\partial_t^2\xi^\Psi_{\frac{1}{2}}+[\xi^\Psi_{\frac{1}{2}},\partial_t^2\xi^\Psi_{\frac{1}{2}}]) \\
&\;=\;\partial_\alpha\mathcal{L}_\Psi(\xi^\Psi_{\frac{1}{2}},\xi^\parallel_{\frac{1}{2}})+|\mathbf{e}_\perp|^2[\xi^\Psi_{\frac{1}{2}},B\partial_l(B\partial_l\xi^\Psi_{\frac{1}{2}})]+B\partial_l|\mathbf{e}_\perp|^2[\xi^\Psi_{\frac{1}{2}},B\partial_l\xi^\Psi_{\frac{1}{2}}] \\
&\quad+\frac{\mathbf{e}_\perp\cdot(2\gamma p\boldsymbol{\kappa}-\rho\mathbf{g})}{1+\gamma\beta}[\xi^\Psi_{\frac{1}{2}},B\partial_l\xi^\parallel_{\frac{1}{2}}] \\
&\quad+\left\{\frac{\rho\mathbf{g}\cdot\mathbf{B}}{B^2(1+\gamma\beta)}\mathbf{e}_\perp\cdot(2B^2\boldsymbol{\kappa}+\rho\mathbf{g})-\mathbf{e}_\perp\cdot\mathbf{g}\mathbf{B}\cdot\nabla_0\rho\right\}[\xi^\Psi_{\frac{1}{2}},\xi^\parallel_{\frac{1}{2}}], \qquad (21)
\end{aligned}
$$

$$
\rho B^2\partial_t^2\xi^\parallel_{\frac{1}{2}}=\mathcal{L}_\parallel(\xi^\Psi_{\frac{1}{2}},\xi^\parallel_{\frac{1}{2}}) \qquad (22)
$$

The above equations recover the nonlinear equations for the filament growth due to line-tied $g$ mode in a shearless slab configuration [8, 9]. It can be seen that effects of magnetic curvature and gravity are similar linearly and nonlinearly.

## III. BALLOONING FILAMENT EQUATIONS IN TOKAMAK FLUX COORDINATES

Next we consider the tokamak equilibrium in a toroidal flux coordinate system $(\Psi_0, \alpha_0, \theta_0)$ so that $\mathbf{B} = \nabla_0\Psi_0 \times \nabla_0\alpha_0 = B^\theta \mathbf{e}_\theta$, where $\alpha_0 = q(\Psi_0)\theta_0 - \zeta_0$, $\theta_0$ and $\zeta_0$ are poloidal and toroidal angles, respectively. Ignoring the $g$-mode, the nonlinear ballooning mode equations are

$$
\begin{aligned}
&\rho|\mathbf{e}_\perp|^2(\partial_\alpha\partial_t^2\xi^\Psi_{\frac{1}{2}} + [\xi^\Psi_{\frac{1}{2}}, \partial_t^2\xi^\Psi_{\frac{1}{2}}]) \\
&= \partial_\alpha \mathcal{L}_\Psi(\xi^\Psi_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) + |\mathbf{e}_\perp|^2[\xi^\Psi_{\frac{1}{2}}, B^\theta\partial_\theta(B^\theta\partial_\theta\xi^\Psi_{\frac{1}{2}})] + B^\theta\partial_\theta|\mathbf{e}_\perp|^2[\xi^\Psi_{\frac{1}{2}}, B^\theta\partial_\theta\xi^\Psi_{\frac{1}{2}}] \\
&\quad + \frac{2\gamma p \mathbf{e}_\perp\cdot\boldsymbol{\kappa}}{1+\gamma\beta}[\xi^\Psi_{\frac{1}{2}}, B^\theta\partial_\theta\xi^\parallel_{\frac{1}{2}}],
\end{aligned}
\tag{23}
$$

$$
\rho B^2 \partial_t^2 \xi^\parallel_{\frac{1}{2}} = \mathcal{L}_\parallel(\xi^\Psi_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}),
\tag{24}
$$

where

$$
\begin{aligned}
\mathcal{L}_\Psi(\xi^\Psi_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) &= B^\theta\partial_\theta(|\mathbf{e}_\perp|^2 B^\theta\partial_\theta\xi^\Psi_{\frac{1}{2}}) + \left[2\mathbf{e}_\perp\cdot\boldsymbol{\kappa}\mathbf{e}_\perp\cdot\nabla_0 p - \frac{4\gamma p(\mathbf{e}_\perp\cdot\boldsymbol{\kappa})^2}{1+\gamma\beta}\right]\xi^\Psi_{\frac{1}{2}} \\
&\quad + \frac{2\gamma p \mathbf{e}_\perp\cdot\boldsymbol{\kappa}}{1+\gamma\beta} B^\theta\partial_\theta\xi^\parallel_{\frac{1}{2}},
\end{aligned}
\tag{25}
$$

$$
\mathcal{L}_\parallel(\xi^\Psi_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) = B^\theta\partial_\theta\left(\frac{\gamma p}{1+\gamma\beta}B^\theta\partial_\theta\xi^\parallel_{\frac{1}{2}} - \frac{2\gamma p\mathbf{e}_\perp\cdot\boldsymbol{\kappa}}{1+\gamma\beta}\xi^\Psi_{\frac{1}{2}}\right).
\tag{26}
$$

The only formal difference here is the choice of $\theta_0$ as the measure of the distance along the equilibrium magnetic field line.

## IV. LARGE ASPECT RATIO TOKAMAK IN SHAFRANOV COORDINATES

For tokamaks with large aspect ratio ($\epsilon = a/R_0 \ll 1$), the configuration may be conveniently represented in the Shafranov coordinates $(r, \theta, \zeta)$ that is defined in the cylindrical coordinate system $(R, Z, \phi)$ as $R = R_0 + r\cos\theta - \Delta(r)$, $Z = r\sin\theta$, $\phi = -\zeta$. Here $R_0$ is the major radius of magnetic axis, $\Delta(r)$ is the Shafranov shift, and we have dropped the "0" subscript in the Lagrangian coordinates ($r_0$,$\theta_0$,$\zeta_0$) for simplicity of notations. Further we transform to the field-line Shafranov coordinate system $(r, \alpha, \theta)$, where $\alpha = q(r)\theta - \zeta$, for the development of the nonlinear ballooning mode equations.

To the second order in $\epsilon = a/R_0$, the tokamak magnetic configuration in the field-line Shafranov coordinate system $(r, \alpha, \theta)$ has the form

$$\mathbf{B} = \frac{I(r)}{R}(\frac{r}{\mathcal{J}} - \frac{1}{R})\mathbf{e}_\alpha + \frac{I(r)r}{q(r)R\mathcal{J}}\mathbf{e}_\theta, \tag{27}$$

where $\mathbf{e}_\alpha = \mathcal{J}\nabla\theta \times \nabla r$, $\mathbf{e}_\theta = \mathcal{J}\nabla r \times \nabla\alpha$, with $\mathcal{J} = Rr(1 - \Delta' \cos\theta)$ being the Jacobian of coordinate transformation. Expansion of equilibrium force balance in $\epsilon$ yields the lowest two orders of Grad-Shafranov equation [13]:

$$\mathcal{O}(\epsilon^0): \qquad \frac{I^2}{qR_0^4}\left(\frac{r^2}{q}\right)' + p' + \frac{II'}{R_0^2} = 0, \tag{28}$$

$$\mathcal{O}(\epsilon\cos\theta): \ \Delta'' + \frac{3-2s}{r}\Delta' + \frac{2}{R_0}\left(2s - 5 - \frac{q^2R_0^2I'}{rI}\right) = 0. \tag{29}$$

Here, $(\cdot)' = d(\cdot)/dr_0$, $s = rq'/q$. The orderings $R_0 \sim \epsilon^{-1}$, $\Delta \sim \Delta' \sim \Delta'' \sim \epsilon$, $I \sim \epsilon^{-2}$, $I' \sim \epsilon^0$ are assumed for a low $\beta$ tokamak in large aspect ratio limit in obtaining (28) and (29).

Shafranov coordinates are not flux coordinates. A common practice was to express $\mathbf{B}\cdot\nabla = B^\theta\partial_\theta$ in flux coordinates, whereas only evaluating equilibrum and metric factors in Shafranov coordinates in the linear ballooning mode operators [13, 14]. This is equivalent to using Shafranov coordinate as an approximate flux coordinate, by keeping only the lowest order of $\mathbf{B}\cdot\nabla$ in $\epsilon$

$$\mathbf{B}\cdot\nabla = B^\alpha\partial_\alpha + B^\theta\partial_\theta = \epsilon\frac{I\Delta'\cos\theta}{R_0^2}\partial_\alpha + \frac{I}{qR_0^2}\partial_\theta \sim \frac{I}{qR_0^2}\partial_\theta \tag{30}$$

Transforming Eqs. (25) and (26) into the above field-line Shafranov coordinate system, we have intermediate nonlinear ballooning equations for tokamak configurations in the large aspect ratio limit

$$\begin{aligned}
&\rho h(\theta)(\partial_\alpha\partial_t^2\xi^r_{\frac{1}{2}} + [\xi^r_{\frac{1}{2}}, \partial_t^2\xi^r_{\frac{1}{2}}]) \\
&= \partial_\alpha\mathcal{L}_r(\xi^r_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) + h(\theta)[\xi^r_{\frac{1}{2}}, B^\theta\partial_\theta(B^\theta\partial_\theta\xi^r_{\frac{1}{2}})] + B^\theta\partial_\theta h(\theta)[\xi^r_{\frac{1}{2}}, B^\theta\partial_\theta\xi^r_{\frac{1}{2}}] \\
&\quad + \frac{2\gamma p g(\theta)}{1+\gamma\beta}[\xi^r_{\frac{1}{2}}, B^\theta\partial_\theta\xi^\parallel_{\frac{1}{2}}],
\end{aligned} \tag{31}$$

$$\rho\partial_t^2\xi^\parallel_{\frac{1}{2}} = \mathcal{L}_\parallel(\xi^r_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}), \tag{32}$$

where $[A, B] \equiv \partial_r A\partial_\alpha B - \partial_\alpha A\partial_r B$, and

$$\begin{aligned}
\mathcal{L}_r(\xi^r_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) &= B^\theta\partial_\theta(h(\theta)B^\theta\partial_\theta\xi^r_{\frac{1}{2}}) + \left[2g(\theta)p' - \frac{4\gamma p g(\theta)^2}{1+\gamma\beta}\right]\xi^r_{\frac{1}{2}} \\
&\quad + \frac{2\gamma p g(\theta)}{1+\gamma\beta}B^\theta\partial_\theta\xi^\parallel_{\frac{1}{2}},
\end{aligned} \tag{33}$$

$$\mathcal{L}_\parallel(\xi^r_{\frac{1}{2}}, \xi^\parallel_{\frac{1}{2}}) = B^\theta\partial_\theta\left(\frac{\gamma\beta}{1+\gamma\beta}B^\theta\partial_\theta\xi^\parallel_{\frac{1}{2}} - \frac{2\gamma\beta g(\theta)}{1+\gamma\beta}\xi^r_{\frac{1}{2}}\right). \tag{34}$$

Here the major ballooning metric factors for a Shafranov equilibrium are evaluated in $(r, \alpha, \theta)$ to $\mathcal{O}(\epsilon^2)$,

$$
\begin{aligned}
g(\theta) = \mathbf{e}_\perp \cdot \boldsymbol{\kappa} = -\frac{\epsilon}{R_0}(\cos\theta + s\theta\sin\theta) \\
-\frac{\epsilon^2 r}{R_0^2}\left[\frac{1}{q^2} - \frac{R_0\Delta'}{r}(1 - \frac{s\theta\sin 2\theta}{2}) - \cos\theta(\cos\theta + s\theta\sin\theta)\right]
\end{aligned}
\quad (35)
$$

$$
\begin{aligned}
h(\theta) = |\mathbf{e}_\perp|^2 = 1 + s^2\theta^2 - 2\epsilon\Delta'[(1 - s^2\theta^2)\cos\theta + s\theta\sin\theta] \\
+\epsilon^2\left[-\left(\frac{r}{qR_0}s\theta\right)^2 + \Delta'^2(1 - s\theta\sin 2\theta + 3s^2\theta^2\cos^2\theta)\right]
\end{aligned}
\quad (36)
$$

Note that the spatial derivatives of the plasma displacement $\boldsymbol{\xi}$ are operated on the normalized coordinate $(r, \alpha, \theta) = (\sqrt{n}r_0, n\alpha_0, \theta_0)$. We thus obtained the $s - \alpha$ model of ballooning filament growth equations in the intermediate nonlinear regime.

## V. SUMMARY

An ideal MHD theory for the filament growth due to a ballooning instability in the intermediate nonlinear regime has been developed in general and toroidal magnetic configurations. The nonlinear equations are further explicitly formulated in a Shafranov coordinate for a circular-shaped tokamak plasma in the limit of large aspect ratio. The equations constitute an $s - \alpha$ model for the intermediate nonlinear ballooning instability.

In this nonlinear regime, there are three major nonlinear effects that are involved in the development of a ballooning filament. From the nonlinear equation for $\xi_\Psi$ in Eq. (23) or Eq. (31), those nonlinear effects are due to radial convection, line bending, and magnetosonic coupling. The physical consequences of each of those nonlinear effects are yet to be explored.

In next step we plan to solve the nonlinear $s - \alpha$ model in Eqs. (31) and (32) both analytically and numerically. We then will compare the solutions with results from direct MHD simulations, using the NIMROD code [15], of nonlinear ballooning instability in a circular tokamak with large aspect ratio. The goal is to figure out how the ballooning filament growth rate and saturation level will depend on the pedestal properties. That will be a step forward in the understanding the precursor and onset phases of ELMs.

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