Linear Resistive Layer Equations in the Presence of Sheared Toroidal Rotation

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Abstract

The linearized resistive magnetohydrodynamics equations are derived for calculations of resistive mode dynamics in the presence of sheared toroidal rotation. The inclusion of sound wave physics is accounted for in the calculation, and hence the equations are valid at arbitrary β. Asymptotic properties of the inner layer equations appropriate for matching to outer region solutions requires the identification of a generalized Mercier criterion which includes the effects of the sheared flow. The marginal stability condition for localized ideal MHD modes is in agreement with the condition derived by M. S. Chu [*Phys. Plasmas*, **5**, 184 (1998)] who employed a different technique. The rotation shear weakens the stabilizing effect of magnetic shear on localized interchanges. Further, the inclusion of sound wave effects introduces the most stringent requirement on the strength of the shear flow to avoid ideal instability.

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I. INTRODUCTION

The effect of sheared toroidal flow on ideal MHD instabilities has been investigated by a number of authors [1, 2, 3, 4, 5, 6]. In particular, Chu[4] examined the effect of shear toroidal rotation on ideal localized interchange modes in arbitrary axisymmetric toroidal geometry. The methodology used to derive this result is based on the variation principal of Frieman and Rotenberg [7] for the ideal MHD stability of equilibria with flow. The calculation was restricted to cases with small toroidal flow but large flow shear in order to capture the destabilizing Kelvin-Helmholtz effect. In this work, we derive the linear resistive layer equations appropriate to the case with sheared toroidal flow. Using the theory of matched asymptotics, the matching data requires the identification of a generalized Mercier criteria that includes the effects of flow shear. Using this technique as a criteria, the results presented here yield precisely the same condition for ideal MHD instability is predicted by Chu [4].

The stability condition for localized interchange instabilities in general toroidal geometry has been known for some time [8]. The stability criterion is characterized by the quantity D_I with instability indicated when $D_I > 0$. The criterion for the resistive MHD version of this mode in general toroidal geometry was derived by the seminal paper by Glasse, Greene and Johnson [9]. This calculation is based on a boundary layer approach where a small region of the plasma encompassing a rational magnetic surface is accounted for. The effect of finite resistivity allows for the reconnection of magnetic field lines. The most general form of the stability condition includes the coupling of pressure driven and current driven contributions to the mode dynamics. In the presence of stabilizing resistive interchange physics, the condition for linear instability becomes exceedingly difficult in tokamak plasmas and generally makes tokamaks linear stable to resistive modes.

In this work, we generalize the Glasser layer calculations by including the effects of sheared toroidal rotation. Careful attention is paid to coupling with the sound waves dynamics. Using match asymptotics, a condition for ideal MHD stability is derived by demanding that the indicial equation which governs the asymptotic equation has real solutions. This uncovers the condition for instability that is consistent with the Chu result [4]. This agreement is identified once a term kept in the Chu analysis is shown to be zero and hence plays no role.

In the following section, the resistive layer equations are derived using the same technique of Glasser et al [9]. In Section III, we examine the asymptotic behavior and derive a generalized ideal interchange criterion.

II RESISTIVE LAYER EQUATIONS

The MHD equilibria state of the plasma is governed by the equations

$$\rho \mathbf{V} \cdot \nabla \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla P, \tag{1}$$

$$-\nabla \Phi + \mathbf{V} \times \mathbf{B} = 0, \tag{2}$$

$$\nabla \cdot (\rho \mathbf{V}) = 0, \tag{3}$$

$$\mathbf{V} \cdot \nabla P = -P \nabla \cdot \mathbf{V}, \tag{4}$$

where $P, \mathbf{J}, \mathbf{B}, \rho$ and Φ represent the plasma pressure, current, magnetic field, mass density and electrostatic potential, respectively. For a toroidal plasma with axisymmetry, the magnetic field can be written

$$\mathbf{B} = I \nabla \zeta + \nabla \zeta \times \nabla \psi \tag{5}$$

where ψ labels the magnetic flux surfaces and ζ is the toroidal angle. The general form for the equilibrium flow profile takes the form

$$\mathbf{V} = \frac{F(\psi)}{\rho} \mathbf{B} + \Omega R^2 \nabla \zeta. \tag{6}$$

For simplicity, we restrict ourselves to cases with only toroidal flow ($F = 0$). In this case, $I = I(\psi)$, and the plasma equilibrium satisfies a generalized Grad-Shafranov equation of the form

$$\Delta * \psi = -II' - \mu_o R^2 \frac{\partial P}{\partial \psi}, \tag{7}$$

where the pressure satisfies

$$P = P_o(\psi) e^{\frac{R^2 \rho_o \Omega^2}{2p_o}}. \tag{8}$$

The exponential factor in Eq. (8) accounts for the rotational forces. The parallel current is given by

$$\frac{\mathbf{J} \cdot \mathbf{B}}{B^2} \equiv \sigma = -\frac{dI}{d\psi} - \mu_o \frac{I}{B^2} \frac{\partial P}{\partial \psi}. \tag{9}$$

A form equivalent to Eq. (5) can be written for the equilibrium magnetic field by introducing the straight-field line variable poloidal angle θ such that

$$\mathbf{B} = q\nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\psi, \tag{10}$$

where $q = q(\psi)$ is the safey factor. It is convenient to introduce the helical angle $\alpha = \zeta - (m/n)\theta$ which corresponds to the resonant angle of the magnetic surface present at $q(\psi_s) = m/n$. The equilibrium magnetic field can be rewritten

$$\mathbf{B} = (q - \frac{m}{n})\nabla\psi \times \nabla\theta + \nabla\alpha \times \nabla\psi \approx q' x \nabla\psi \times \nabla\theta + \nabla\alpha \times \nabla\psi \tag{11}$$

where the second approximate form is used in the vicinity of the rational surface with $q' = dq/d\psi(\psi = \psi_s)$ and $x = \psi - \psi_s$.

The calculation that follows uses a conventional methodology for deriving resistive layer equations [9, 10]. In particular, a narrow layer of width $\epsilon \ll 1$ at the magnetic surface ψ_s where $q(\psi_s) = m/n$ is rational is considered for the investigation of slowly growing modes. The standard resistive MHD layer ordering is used for the growth rate γ and extent and magnetic surface label $x \equiv \psi - \psi_o$,

$$\frac{x}{\psi_s} \sim \gamma\tau_a \sim S^{-1/3} = \epsilon \qquad (12)$$

where τ_a is the characteristic ideal MHD timescale and $S = \tau_R/\tau_a \sim \eta$ is the ratio of the resistive diffusion time across the toroidal cross-section to the ideal MHD time. Fluctuating quantities vary rapidly in x, such that $\partial\tilde{f}/\partial x \sim \epsilon^{-1}\tilde{f}/\psi_s$ while equilibrium quantities vary slowly in the vicinity of the rational surface, $\partial F/\partial x \sim F/\psi_s$.

All subsequent quantities are written as functions of the three independent variables x, θ and α. With an axisymmetric equilibria, linear fluctuating quantities can be written

$$\tilde{f} = f(x,\theta)e^{\gamma t - in\alpha}. \qquad (13)$$

We define a "field line" averaging operator at fixed x and α given by

$$< f >= \frac{\oint \frac{d\theta}{\mathbf{B}\cdot\nabla\theta} f(x,\theta,\alpha)}{\oint \frac{d\theta}{\mathbf{B}\cdot\nabla\theta}} = \frac{\oint \frac{d\theta}{2\pi}\sqrt{g}f}{\hat{V}'}, \qquad (14)$$

where $\sqrt{g} = 1/\nabla\psi \times \nabla\theta \cdot \nabla\zeta$ is the Jacobian and $\hat{V}' = \oint d\theta\sqrt{g}/2\pi$. Field line averages of fluctuating quanties satisfy $< \tilde{f} >= e^{-in\alpha} < f > (x)$ while the field line average of an equilibrium quantity is soley a function of x.

As in Ref. [4], we consider the flow amplitude to be small but the flow shear to be order unity. As such, the rotational force correction to the pressure profile in Eq. (8) will not be considered. We note that the effect of finite rotation on Mercier stability has been analyzed for high aspect ratio, low β plasmas [5, 6]. In the following, no restrictions are placed on aspect ratio or plasma β. Flow shear terms typically appear as part of the advective derivative of the form

$$\mathbf{V}\cdot\nabla\tilde{f} = -in\Omega\tilde{f} = -in\Omega' x\tilde{f}, \qquad (15)$$

where $\Omega' = d\Omega/d\psi(\psi = \psi_s)$. This implies the ordering $\mathbf{V}\cdot\nabla\tilde{f} \sim \epsilon\tilde{f}/\tau_a$.

The linearized resistive MHD force balance, induction equation and pressure evolution are given respectively by

$$\rho\frac{\partial\tilde{\mathbf{v}}}{\partial t} + \rho\mathbf{V}\cdot\nabla\tilde{\mathbf{v}} + \rho\tilde{\mathbf{v}}\cdot\nabla\mathbf{V} = \tilde{\mathbf{j}}\times\mathbf{B} + \mathbf{J}\times\tilde{\mathbf{b}} - \nabla\tilde{p}, \tag{16}$$

$$\frac{\partial\tilde{\mathbf{b}}}{\partial t} - \nabla\times(\tilde{\mathbf{v}}\times\mathbf{B}) - \nabla\times(\mathbf{V}\times\tilde{\mathbf{b}}) = \frac{\eta}{\mu_o}\nabla^2\tilde{\mathbf{b}}, \tag{17}$$

$$\frac{\partial\tilde{p}}{\partial t} + \mathbf{V}\cdot\nabla\tilde{p} + \tilde{\mathbf{v}}\cdot\nabla P + \frac{5}{3}P\nabla\cdot\tilde{\mathbf{v}} = 0, \tag{18}$$

where η is the plasma resistivity. We write the perturbed magnetic field and flow as

$$\mathbf{b} = b^\psi\frac{\nabla\psi}{|\nabla\psi|^2} + b_\perp\frac{\mathbf{B}\times\nabla\psi}{B^2} + b_{||}\frac{\mathbf{B}}{B^2}, \tag{19}$$

$$\mathbf{v} = v^\psi\frac{\nabla\psi}{|\nabla\psi|^2} + v_\perp\frac{\mathbf{B}\times\nabla\psi}{B^2} + v_{||}\frac{\mathbf{B}}{B^2}. \tag{20}$$

Consistent ordering requires that these terms are expanded in orders of ϵ with $b^\psi/b_\perp \sim b^\psi/b_{||} \sim \epsilon$ and $v^\psi/v_\perp \sim v^\psi/v_{||} \sim \epsilon$ to leading order.

The leading order contributions to the $\nabla\psi$ and $\mathbf{B}\times\nabla\psi/|\nabla\psi|^2$ projections of the induction equation and the pressure equations are

$$\frac{1}{\sqrt{g}}\frac{\partial v^\psi}{\partial\theta} = \frac{1}{\sqrt{g}}\frac{\partial v_\perp}{\partial\theta} = \frac{5}{3}P\nabla\cdot\mathbf{v} = 0, \tag{21}$$

which yields solutions $v^\psi =< v^\psi >$, $v_\perp =< v_\perp >$ and

$$v_{||} =< v_{||} > \frac{B^2}{< B^2 >} + v_\perp\frac{B^2}{\mu_o P'}(\sigma - \frac{< \sigma B^2 >}{< B^2 >}) \tag{22}$$

where $P' = dP/d\psi(\psi = \psi_s)$ and Eq. (9) is used. Projections of the momentum balance equations along $\mathbf{B}$ and in the $\nabla\psi$ direction yield the leading order solutions

$$\frac{1}{\sqrt{g}}\frac{\partial p}{\partial\theta} = |\nabla\psi|^2\frac{\partial}{\partial x}(\mu_o p + b_{||}) = 0, \tag{23}$$

which yields $p =< p >$ and $b_{||} = -\mu_o p$.

From the condition $\nabla \cdot \tilde{\mathbf{b}} = 0$, one can derive

$$< \frac{\partial b^\psi}{\partial x} > + in < b_\perp >= 0, \tag{24}$$

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial \theta} (\frac{b_{||} + I b_\perp}{B^2}) = - \frac{\partial b^\psi}{\partial x} - in b_\perp. \tag{25}$$

A perturbed quasineutrality equation can be obtained by considering the operation $\nabla \cdot [(\mathbf{B} B^{-}2) \times$ momentum equation]. To leading order this becomes

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial \theta} (\frac{\partial b_\perp}{\partial x} \frac{|\nabla \psi|^2}{B^2} + \mu_o \frac{I}{B^2} \frac{\partial p}{\partial x}) = 0, \tag{26}$$

where Ampere's law has been used. This allows one to write the solution

$$\frac{\partial b_\perp}{\partial x} =< \frac{\partial b_\perp}{\partial x} > \frac{\frac{B^2}{|\nabla \psi|^2}}{< \frac{B^2}{|\nabla \psi|^2} >} + \frac{\partial p}{\partial x} \frac{1}{P'} (\frac{\sigma B^2}{|\nabla \psi|^2} - \frac{B^2}{|\nabla \psi|^2} \frac{< \frac{\sigma B^2}{|\nabla \psi|^2} >}{< \frac{B^2}{|\nabla \psi|^2} >}). \tag{27}$$

To this point, the derivation of the layer equations is exactly the same as prior calculations [9] since the flow shear has not entered to this order. To derive the appropriate layer equations, we need to go to higher order in ϵ. This procedure generates order ϵ corrections to various fluctuating parameters which can be annihilated using the field line averaging operator, Eq. (14).

The relevant averaged $\nabla \psi$ projection of the induction equation is given by

$$(\gamma - in\Omega' x) < b^\psi > + \frac{inq'x}{\hat{V}'} v^\psi = \frac{\eta}{\mu_o} (< \frac{\partial^2 b^\psi}{\partial x^2} > - in \frac{dp}{dx} \frac{q' H}{P' \hat{V}'} \frac{< B^2 >}{< \frac{B^2}{|\nabla \psi|^2} >}), \tag{28}$$

where Eq. (27) is used and we introduce the factor H as in Ref. [9],

$$H = \frac{\hat{V}'}{q'} < \frac{B^2}{|\nabla \psi|^2} > (\frac{< \sigma B^2 >}{< B^2 >} - \frac{< \frac{\sigma B^2}{|\nabla \psi|^2} >}{< \frac{B^2}{|\nabla \psi|^2} >}). \tag{29}$$

The last term in Eq. (28) accounts for perturbed Pfirsch-Schlüter current effects. The quantity H asymptotes to zero at infinite aspect ratio with circular flux surfaces.

The equivalent of a vorticity or quasineutrality equation can be derived by applying the same operator as used to obtain Eq. (26) from the momentum balance equation. We proceed by field line averaging the equation obtained at one higher order in ϵ and obtain

$$\frac{\rho - mu_o}{in}(\gamma - in\Omega' x) < \frac{|\nabla\psi|^2}{B^2} > \frac{\partial^2 v^\psi}{\partial x^2} + \rho\mu_o \frac{\partial}{\partial x}(\Omega' x v_\perp) < \frac{\sigma}{P'}\mathbf{B}\cdot\nabla R^2 >$$

$$= -\frac{q' x}{\hat{V}' < \frac{B^2}{|\nabla\psi|^2} >} < \frac{\partial^2 b^\psi}{\partial x^2} > + \frac{inq' x}{P'}\frac{\partial p}{\partial x}(< \frac{\sigma}{\sqrt{g}} > - \frac{< \frac{\sigma B^2}{|\nabla\psi|^2} >}{\hat{V} < \frac{B^2}{|\nabla\psi|^2} >})$$

$$+ in\mu_o p(2\mu_o P' < \frac{1}{B^2} > - < \frac{\mathbf{B}\times\nabla\alpha\cdot\nabla B^2}{B^4} >) - < \sigma\frac{\partial b^\psi}{\partial x} > - in < \sigma b_\perp >$$

$$+ < \frac{\partial p^{(1)}}{\partial x}\mathbf{B}\cdot\nabla\frac{\sigma}{P'} > . \tag{30}$$

The problematic term is the last one since it involves order ϵ corrections to the perturbed pressure. An equation to cancel this term can be obtained by multiplying the parallel momentum balance equation with the the factor σ/P' and then averaging over the field line and taking an x derivative. This procedure yields the equation

$$\rho\frac{\partial}{\partial x}(\gamma - in\Omega' x) < \frac{\sigma}{P'}v_{||} > + \rho\frac{\partial}{\partial x}(\Omega' x v_\perp) < \frac{\sigma}{P'}\mathbf{B}\cdot\nabla R^2 > + \rho I\Omega'\frac{\partial v^\psi}{\partial x} < \frac{\sigma}{P'} >$$

$$= - < \sigma\frac{\partial b^\psi}{\partial x} > + inq' p < \frac{\sigma}{P'\sqrt{g}} > + inq' x\frac{\partial p}{\partial x} < \frac{\sigma}{P'\sqrt{g}} >$$

$$- < \frac{\sigma}{P'}\mathbf{B}\cdot\nabla\frac{\partial p^{(1)}}{\partial x} > . \tag{31}$$

Additionally, we also need the averaged parallel momentum balance equation to leading non-trivial order. This is given by

$$\rho(\gamma - in\Omega' x) < v_{||} > + \rho I\Omega' v^\psi = - < b^\psi > P' + \frac{inq' x}{\hat{V}'}p, \tag{32}$$

where the effect of the shear flow enters on the left side. Combinations of Eqs. (29)-(32), as well as the defining the variable quantity Γ by

$$\Gamma = \frac{\hat{V}'^2}{q'^2} < \frac{B^2}{|\nabla\psi|^2} > (-\mu_o^2 p P' < \frac{1}{B^2} > + < b_\perp > \frac{< \sigma B^2 >}{< B^2 >} - < \sigma b_\perp >), \tag{33}$$

and the equilibrium quantities

$$E = \frac{\mu_o P' \hat{V}'^2}{q'^2} < \frac{B^2}{|\nabla\psi|^2} > (-\frac{\hat{V}''}{\hat{V}'} + \frac{q' I}{\hat{V}' < B^2 >}), \tag{34}$$

$$M = < \frac{B^2}{|\nabla\psi|^2} > (< \frac{|\nabla\psi|^2}{B^2} > + \frac{1}{\mu_o^2 P'^2} < \sigma^2 B^2 > - \frac{1}{\mu_o^2 P'^2} \frac{< \sigma B^2 >^2}{< B^2 >}), \tag{35}$$

allows one to arrive at the desired equation

$$\rho\mu_o \frac{\hat{V}'^2}{n^2 q'^2} (\gamma - in\Omega' x) M \frac{\partial^2 v^\psi}{\partial x^2} = -\frac{i\hat{V}'}{nq'} x < \frac{\partial^2 b^\psi}{\partial x^2} > - \frac{x}{P'} H \frac{\partial p}{\partial x} - \Gamma - p\frac{E}{P'}. \tag{36}$$

Here, the quantity E contains the effect of an average magnetic well ($\hat{V}'' < 0$) or hill ($\hat{V}'' > 0$). This plays an important role in determining the interchange drive.

An equation for the variable Γ can be derived by computing the x derivative of Eq. (33) and using Eqs. (24), (27) and (29). This leads to the equation

$$\frac{\partial \Gamma}{\partial x} = -\frac{F}{P'} \frac{\partial p}{\partial x} + \frac{i\hat{V}'}{nq'} H < \frac{\partial^2 b^\psi}{\partial x^2} >, \tag{37}$$

where the equilibrium quantity F is defined by [9]

$$F = \frac{\hat{V}'^2}{q'^2} < \frac{B^2}{|\nabla\psi|^2} > (< \frac{\sigma^2 B^2}{|\nabla\psi|^2} > - \frac{< \frac{\sigma B^2}{|\nabla\psi|} >^2}{< \frac{B^2}{|\nabla\psi|^2} >} + \mu_o^2 P'^2 < \frac{1}{B^2} >). \tag{38}$$

A common approximation is to integrate the above equation once in x and assume that the integration constant is zero. This leads to the equation $\Gamma = -(F/P')p + H(i\hat{V}'/nq') < \partial b^\psi/\partial x >$ which can be used in Eq. (36).

The final layer equation describes the evolution of the perturbed pressure or perturbed parallel magnetic field. From the parallel induction equation to next order in ϵ, one derives

$$(\gamma - in\Omega' x)b_{||} - I\Omega' < \frac{b^\psi}{B^2} > + inq'x < \frac{v_{||}}{\sqrt{g}B^2} >$$

$$+ v^\psi(\mu_o P' < \frac{1}{B^2} > + < \frac{\nabla\psi \cdot \nabla B^2}{B^2|\nabla\psi|^2} >) - \frac{\eta}{\mu_o} < \frac{|\nabla\psi|^2}{B^2} > \frac{\partial^2 b_{||}}{\partial x^2}$$

$$= - < v_\perp^{(1)} \mathbf{B} \cdot \nabla \frac{\sigma}{\mu_o P'} > - < \nabla \cdot \mathbf{v}^{(1)} >, \qquad (39)$$

where the terms on the right side account for order ϵ corrections to $v_\perp$ and the plasma compressibility, respectively. The first term can be eliminated by constructing the field line average of the product of σ/P' times the $\mathbf{B} \times \nabla\psi/|\nabla\psi|^2$ projection of the induction equation. This produces the equation

$$(\gamma - in\Omega' x)\frac{< \sigma b_\perp >}{P'} + \frac{\Omega'}{P'} < \sigma b^\psi > + inq'x < \frac{\sigma v_\perp}{P'\sqrt{g}} > + v^\psi < \frac{\sigma B^2}{P'|\nabla\psi|^2} s >$$

$$- \frac{\eta}{\mu_o} < \frac{\sigma|\nabla\psi|^2}{P'} \frac{\partial^2 b_\perp}{\partial x^2} > = < \frac{\sigma}{P'} \mathbf{B} \cdot \nabla v_\perp^{(1)} >, \qquad (40)$$

where $s = \mathbf{B} \times \nabla\psi \cdot \nabla \times (\mathbf{B} \times \nabla\psi)/B^2|\nabla\psi|^2$ is the local shear. Combining Eqs. (39) and (40) and using Eqs. (23), (27), (28), (33), (34) and (35) produces the equation

$$\frac{q'^2}{\mu_o P' \hat{V}'^2 < \frac{B^2}{|\nabla\psi|^2} >}[(\gamma - in\Omega' x)\Gamma + Ev^\psi] - \frac{I\Omega'}{< B^2 >} < b^\psi >$$

$$+ \frac{inq'x}{\hat{V}'} \frac{< v_{||} >}{< B^2 >} + \eta \frac{\partial^2 p}{\partial x^2} \frac{M}{< \frac{B^2}{|\nabla\psi|^2} >} = - < \nabla \cdot \mathbf{v}^{(1)} >, \qquad (41)$$

where the compressibility term remains to be eliminated. This term also arises in the next order pressure evolution equation given by

$$(\gamma - in\Omega' x)p + v^\psi P' + \frac{5}{3} P < \nabla \cdot \mathbf{v}^{(1)} > = 0 \qquad (42)$$

Combining Eqs. (41) and (42) yields the required equation for the perturbed pressure once Eq. (28) is incorporated.

Four equations for the variables $< b^\psi >$, v^ψ, p and Γ are derived above. In the following we use the normalized variables introduced in Ref. [9]

$$v^\psi \equiv (\gamma - in\Omega' x)\xi, < b^\psi > \equiv -\frac{inq'}{\hat{V}'} L_R \Psi, p \equiv -P'\Upsilon. \tag{43}$$

Additionally, the standard definitions for the normalized time and length scales are given by

$$Q = \gamma\tau_R, X = x/L_R, \tag{44}$$

where

$$\tau_R \equiv \frac{\hat{V}'^{2/3}\rho^{1/3}\mu_o^{1/3}M^{1/3} < B^2/|\nabla\psi|^2 >^{1/3}}{(\eta/\mu_o)^{1/3}n^{2/3}q'^{2/3} < B^2 >^{1/3}}. \tag{45}$$

$$L_R \equiv \frac{\hat{V}'^{1/3}\rho^{1/6}\mu_o^{1/6}M^{1/6}(\eta/\mu_o)^{1/3} < B^2 >^{1/3}}{n^{1/3}q'^{1/3} < B^2/|\nabla\psi|^2 >^{1/3}}. \tag{46}$$

Additionally, we introduce a normalized shear flow M_a which is consistent with the definition used in Chu's paper [4]

$$M_a \equiv n\Omega'\tau_R L_R = \Omega' \frac{\hat{V}'}{q'}\sqrt{\rho\mu_o M}. \tag{47}$$

In this definition the shear of the flow is normalized to the product of the magnetic shear and the Alfven speed.

Using these definitions, the four layer equations are given by

$$(Q - iM_a X)(\Psi - x\xi) = \Psi_{XX} - H\Upsilon_X, \tag{48}$$

$$\Gamma_X = F\Upsilon_X + H\Psi_{XX}, \tag{49}$$

$$\frac{d}{dX}(Q - iM_a X)^2 \frac{d\xi}{dX} + X\Psi_{XX} + \Gamma + E\Upsilon - HX\Upsilon_X = 0, \tag{50}$$

$$\frac{\Upsilon_{XX}}{Q - iM_a X} + \frac{X\Psi - X^2\Upsilon}{(Q - iM_a X)^2} - K\Gamma + (G - KE)\xi - G\Upsilon$$

$$+\frac{iM_p}{Q - iM_a X}(X\xi - \Psi) = 0, \tag{51}$$

where we use the standard definitions [9]

$$K = \frac{q'^2}{\mu_o^2 P'^2 \hat{V}'^2 M} \frac{< B^2 >}{< \frac{B^2}{|\nabla\psi|^2} >}, \tag{52}$$

$$G = \frac{< B^2 >}{M \frac{5}{3} \mu_o P}. \tag{53}$$

The term M_p in Eq. (51) is proportional to the flow shear as well and is defined by

$$M_p = \Omega' \frac{I}{P'} \sqrt{\frac{\rho}{\mu_o M}} = M_a \frac{q' I}{\mu_o P' M \hat{V}'} \tag{54}$$

Equation (48) is the radial induction equation with the left side denoting the ideal MHD contributions including the role of the shear flow. The right side of (48) contains the resistive contributions to the induction equations and accounts for toroidal Pfirsch-Schlüter currents through the term H. Equation (49) is the same as Eq. (37). Equation (50) is essentially the quasineutrality equation with the first term accounting for polarization currents, the second term accounting for the $\mathbf{B} \cdot \nabla(J_{||}/B)$ term and the remaining terms accouting for pressure/curvature effects. Equation (51) accounts for the pressure evolution with the first term accounting for resistive diffusion of the perturbed parallel magnetic field. The second term accounts for forces trying to equilibrate the pressure along the perturbed field lines. The remaining terms account for sound wave effects as well as containing contributions from the pressure/curvature and shear flow instability drives.

Equation (49) can be integrated once in X to obtain an equation for Γ. Assuming that the integration constant can be ignored, Eqs (50) and (51) can be simplified.

$$\frac{d}{dX}(Q - iM_a X)^2 \frac{d\xi}{dX} + X\Psi_{XX} + (E + F)\Upsilon + H(\Psi_X - X\Upsilon_X) = 0, \tag{55}$$

$$\frac{\Upsilon_{XX}}{Q - iM_a X} + \frac{X\Psi - X^2\Upsilon}{(Q - iM_a X)^2} - KH\Psi_X + (G - KE - KF)\xi - G\Upsilon$$
$$+ \frac{iM_p}{Q - iM_a X}(X\xi - \Psi) = 0. \tag{56}$$

III. Asymptotic Properties

The calculation of resistive stability requires the asymptotic matching of the exterior limit of the inner resistive layer solutions to the inner limit of the ideal MHD solutions. In the limit of zero shear flow, this asymptotic behavior is characterized by the eigenfuction shape

$$\xi = (|x|^{\alpha_l} + \Delta |x|^{\alpha_s}) \tag{57}$$

where the "large" and "small" Mercier parameters are given by

$$\alpha_{l,s} = -\frac{1}{2} \mp \sqrt{-D_I} = -\frac{1}{2} \mp \sqrt{\frac{1}{4} - E - F - H}, \tag{58}$$

and the matching data is characterized by the parameter Δ. Ideal MHD stability is violated when D_I approaches zero. For positive D_I, oscillitory solutions are indicated as the rational surface is approached.

Noting that the asymptotic behavior of the layer equations with shear flow should also indicate the generalized Mercier criterion, we use this technique to derive a condition for the onset of localized interchange modes in the presence of shear flow. To accomplish this, we examine the behavior of Eqs. (48), (55), and (56) at large X. The radial induction equation simplifies to

$$\Psi - X\xi \approx 0, \tag{59}$$

which is also the condition in the absence of shear flow. The quasineutrality condition at large X becomes

$$-M_a^2 \frac{d}{dX} X^2 \frac{d\xi}{dX} + X\Psi_{XX} + (E + F)\Upsilon + H(\Psi_X - X\Upsilon_X) \approx 0, \tag{60}$$

where the shear flow term enters through the polarization current. Using Eq. (59), the first two terms combine to same mathematical structure and Eq. (60) can be rewritten

$$(1 - M_a^2)\frac{d}{dX}X^2\frac{d\xi}{dx} + (E+F)\Upsilon + H[\frac{d}{dX}(X\xi) - X\Upsilon_X] \approx 0, \tag{61}$$

The first term represents the stabilizing effect of magnetic field line bending. Equation (61) indicates that one of the effects of shear flow is to counteract this stabilizing effect [3, 4]. In particular, an apparent resonance occurs when $|M_a|$ approaches unity where this differential equation becomes singular. However, once sound wave effects are accounted for in the pressure evolution equation, we will find that a more restrictive condition on the amplitude of M_a.

In the absense of shear flow, the large X solution to the pressure equation is $\Upsilon \approx \xi$, which describes the pressure perturbation is due to the advection of the pressure by the ideal MHD displacement. However, Eq. (56) also includes sound wave effects which become important in the asymptotic region when shear flow is present. In particular, the pressure equation at large X becomes

$$\Upsilon \approx \xi + \frac{(E+F)K\xi + HK\Psi_X}{\frac{1}{M_a^2} - G - FK}, \tag{62}$$

which can be rewritten

$$\Upsilon \approx \xi + \frac{K\beta_{5/3}M_a^2}{\beta_{5/3} - M_a^2}[(E+F)\xi + H\Psi_x] \tag{63}$$

where the quantity $\beta_{5/3}$ is defined by

$$\beta_{5/3} = \frac{1}{G+FK} = \frac{\frac{5}{3}\mu_o P}{< B^2 > /M + \frac{5}{3}\mu_o PFK}. \tag{64}$$

The sound wave coupling brings into a resonance when $M_a^2 = \beta_{5/3}$. This is the same resonance condition found by Chu [4] (a slightly different notation is used here). Since $\beta_{5/3} < 1$, this condition is more restrictive that the condition $M_a^2 = 1$.

Combining Eqs. (61) and (62), we derive an equation governing the asymptotic behavior of the linearized eigenfunction in the presence of shear flow.

$$(1 - M_a^2 - H^2 \frac{K M_a^2 \beta_{5/3}}{\beta_{5/3} - M_a^2}) \frac{d}{dX} X^2 \frac{d\xi}{dX} + \xi(E + F + H)[1 + (E + F) \frac{K \beta_{5/3} M_a^2}{\beta_{5/3} - M_a^2}]. \tag{65}$$

Note that the sound wave resonance enters in two ways. The "field line bending" term is modified by a term proportional to H^2. This response is totally a toroidal effect and is due to the pressure response needed to account for the Pfirsch-Schlüter effect. The shear flow also modifies the pressure curvature drive term. In analogy with the Equation (57), Equation (61) satisfies

$$\xi \approx (|x|^{\alpha_l} + \Delta |x|^{\alpha_s}), \tag{66}$$

where the Mercier indices are modifed to be $\alpha_{l,s} = -1/2 \mp \sqrt{-D_{I,\Omega}}$ where

$$D_{I,\Omega} = (E + F + H) \frac{1 + (E + F) \frac{K M_a^2 \beta_{5/3}}{\beta_{5/3} - M_a^2}}{1 - M_a^2 - H^2 \frac{K M_a^2 \beta_{5/3}}{\beta_{5/3} - M_a^2}} - \frac{1}{4}. \tag{67}$$

Using $D_{I,\Omega} = 0$ as a marginal stability condition and assuming $\beta_\Gamma > M_a^2$, we find the stability condition to be

$$E + F + H - \frac{1}{4} + \frac{M_a^2}{4} + \frac{\beta_{5/3} M_a^2 K}{\beta_{5/3} - M_a^2} (E + F + \frac{H}{2})^2 < 0. \tag{68}$$

This condition is in agreement with Chu [4] when you account for the fact that a term Chu includes in his expression is exactly zero. This is shown in the appendix.

It is interesting to note that condition (68) yields a non-trivial restriction on ideal MHD stability in the limit of zero equilibrium pressure gradient

$P' = 0$. Taking the $P' \to 0$ limit, we see that the stability condition becomes

$$-\frac{1}{4} + \frac{M_a^2}{4} + \frac{\beta_{5/3} M_a^2}{\beta_{5/3} - M_a^2} C^2 < 0 \tag{69}$$

where the coupling coefficient C is given by

$$C^2 \equiv \lim_{P' \to 0} K(E + F + \frac{H}{2})^2$$

$$= \frac{\hat{V}'^2 < B^2 >< \frac{B^2}{|\nabla\psi|^2} >}{M q'^2} [-\frac{\hat{V}''}{\hat{V}'} + \frac{I q'}{2\hat{V}' < B^2 >} + \frac{I q' < \frac{1}{|\nabla\psi|^2} >}{2\hat{V}' < \frac{B^2}{|\nabla\psi|^2} >}]^2, \tag{70}$$

which in large aspect ratio tokamak ordering with $q = q(r)$ becomes [11]

$$C^2 \approx \frac{4}{R^2 |\nabla q|^2} \frac{[q^2 - 1 + \frac{q^3}{2r^3}\frac{dq}{dr}\int_0^r dr' \frac{r'^3}{q(r')^2}]^2}{1 + 2q^2}. \tag{71}$$

This expression becomes approximately $C^2 \approx 4/(R^2|\nabla q|^2)$ in reversed field pinch ordering.

IV Discussion

Properties of the linear resistive layer equations are used to derive the ideal MHD stability condition to localized interchange modes in the presence of sheared toroidal flows. The results of the calculation produces a modified Mercier criterion, Eq. (67). The marginal stability condition indicated by $D_{I,\Omega} = 0$ is in agreement with the calculation of Chu [4] who used a variational principal to derive his result. This marginal stability condition is relevant when a condition on the shear flow given by $M_a^2 < \beta_\Gamma$ is satisfied. For shear flows that exceed this limit produced from coupling to sound wave physics, a Kelvin-Helmholz drive would excite ideal MHD instabilities rather than pressure/curvature effects.

In addition to addressing the ideal MHD properties, another application of these linearized equations is to determine the effect of sheared toroidal flow on resistive stability. There have been previous efforts to address the role of sheared flow on tearing instability [12, 13, 14, 15, 16]. However, what is missing in these efforts is the role of the sound wave coupling. In particular, it has been noted that in the absence of the sound wave physics, As indicated by the linear analysis, this can have an important practical effect at low plasma β. We leave calculations of the resistive stability analysis for subsequent work.

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Appendix: Demonstration that $A = 0$

Criteria (68) differs from Chu's expression by the factor A. We can show that this term is precisely zero and therefore does not affect the final result. The important aspect of the A is that is proportional to the term $< \hat{z}\cdot\nabla V >$. Noting that $V = V(\psi)$ and $\hat{z} = \nabla R^2 \times \nabla\zeta/2$, this term becomes

$$< \hat{z}\cdot\nabla V >= \frac{V'}{2} < \nabla R^2 \times \nabla\zeta \cdot \nabla\psi >=< \mathbf{B}\cdot\nabla(\frac{R^2 V'}{2}) >= 0$$

where the fact that R is independent of torodial angle ζ is used as well as the property that the field line averaging operation annihilates derivatives along the equilibrium magnetic field.

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